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Long-run investment under uncertain demand

Luca Di Corato* Michele Moretto† Sergio Vergalli‡

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Abstract

In the literature investigating the impact of uncertainty on short-run and long-run investment, the most part of authors have used a log linear profit function. This functional form has been generally considered a reasonable approximation for more general ones and has the advantage of providing closed form solutions for both short-run investment rule and long-run rate of capital accumulation. In this paper, we consider a firm facing a linear demand function with additive shocks and present a technique for the analytical approximation of the long-run average rate of capital accumulation for the case of an inverted U-shape profit function. We then compare the long-run rates of capital accumulation calculated under both assumptions within a plausible range of parameter values. We notice significant differences and conclude that the choice of a log linear functional form has a non-trivial impact on the magnitude of the long run rate of capital accumulation.

KEYWORDS: Investment, demand uncertainty, irreversibility.
JEL CLASSIFICATION: C61, D92, E22.

1 Introduction

In this paper, we compare the long-run average rate of capital accumulation determined under the assumption of a firm facing an isoelastic demand function with multiplicative shocks with the rate calculated under the assumption of a linear demand function with additive shocks.

In the literature on investment under uncertainty, the most part of scholars, use an isoelastic profit function where shocks enter multiplicatively (i.e., a log linear profit function).[1] This form is consistent, for instance, with a firm facing an isoelastic demand function with multiplicative shocks and a Cobb-Douglas production technology. The use of a log linear profit function has the advantage of leading to closed form solutions for both the short-run optimal investment rule and the long-run average rate of capital accumulation (see for instance Hartman and Hendrickson, 2002).[2] Up to our readings, only few authors[3] use a linear demand function with shocks entering additively. In this case, handling the corresponding inverted U-shape profit function is relatively straightforward when it comes to the derivation of a closed form solution for the short-run optimal investment rule. In contrast, when investigating the long-run rate of capital accumulation, it turns out to be problematic in that an analytical solution does not exist.

This paper has two aims. First, we derive and present an analytical approximation of the short-run investment rule under the assumption of a linear demand function. We then show how to use such approximation in order to derive the corresponding 1) steady-state distribution of the optimal stock of capital and 2) the long-run average rate of capital accumulation. Second, we want to compare the long-run rates of

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[1] See among others Bentolila and Bertola (1990), Caballero (1991), Bertola and Caballero (1994), Dixit (1995), Abel and Eberly (1997), Bertola (1998) and Guo and Pham (2005). We can add also the papers by Grenadier (2002) and Back and Paulsen (2009), dealing with strategic investment in capital stocks under uncertainty.

[2] Hartman and Hendrickson (2002) derive the optimal investment strategy for a risk-neutral firm under partial reversibility, stochastic capital prices and random shocks to the capital's marginal revenue product. The authors then examine the effects of increased uncertainty on both optimal investment strategy and long-run average growth rate of capital.

[3] See for instance Pindyck (1988), Dangl (1999) , Sarkar (2009), and Baldursson (1998) for an oligopoly.

capital accumulation determined under both assumptions, i.e., isoelastic vs. linear demand. We do it within a plausible range of parameter values and notice that there may be significant differences. This allows us to conclude that the choice of a log linear functional form has a non-trivial impact on the magnitude of the long run rate of capital accumulation.

The structure of this paper is as follows. In Section 2, we introduce a general model of irreversible investment under uncertainty. Section 3 presents the long-run average rate of capital accumulation for the case of an isoelastic demand function with multiplicative shocks and for the case of a linear demand with additive shocks. In Section 4 we compare the rates obtained under both assumptions. Section 5 concludes.

2 The Basic Model

Let's start by modeling a general firm's investment problem. Consider a risk-neutral firm that costlessly produces a flow of non-storable goods (or services). The firm is able to produce the good in proportion of its capacity and sell it in a market at the instantaneous price p_t. More specifically, we assume that inverse demand takes the following form:[4]

$$\begin{aligned} p_t &= p(K_t, Y_t) \\ \partial p(K_t, Y_t)/\partial K_t &< 0,\ \partial p(K_t, Y_t)/\partial Y_t > 0 \end{aligned} \tag{1}$$

where K_t is the firm's capacity[5] and Y_t is an index shifting the direct demand curve according to the following diffusion process:

$$d\ln[Y_t] = [\alpha - (1/2)\sigma^2]dt + \sigma dW_t \tag{2}$$

where α and σ are the drift and volatility parameter for Y_t and dW_t is the increment of the standard Wiener process satisfying $E[dW_t] = 0$ and $E[dW_t^2] = dt$.

The total instantaneous profit to a firm with capital, K_t, is given by:

$$\Pi(K_t, Y_t) = p(K_t, Y_t)K_t \tag{3}$$

The firm may increase its capacity at any time $t > 0$. Investment is irreversible and each unit of capital costs $c > 0$. Hence, ignoring for simplicity depreciation,[6] the firm's capital stock expansion at each t is given by $dK_t \geq 0$.

Let's now determine the optimal investment policy to be set by the firm. This entails the optimal definition of two interrelated issues: how much to invest, and when.[7] Such policy is defined by maximizing the expected present value of its net benefits subject to (1) and (3) and given the initial values $Y_0 = Y > 0$ and $K_0 = K > 0$. Formally, the risk- neutral firm must solve the following problem:[8]

$$\begin{aligned} V(K_t, Y_t) &= \max_{K_t} E_0[\int_0^\infty e^{-rt}\Pi(K_t, Y_t)dt - cdK_t \mid K_0 = K,\ Y_0 = Y] \\ \text{s.t. } dK_t &\geq 0 \text{ and (2) for all } t \end{aligned} \tag{4}$$

where $r > 0$ is the discount rate and $E_0[.]$ is the expectation operator.[9]

By applying a standard dynamic programming approach, the firm's value function, $V(K_t, Y_t)$, solves the following differential equation:

$$rV(K_t, Y_t)dt = \Pi(K_t, Y_t)dt + E_0[dV(K_t, Y_t)] \tag{5}$$

[4] Note that the general form given in (1) may include both additive or multiplicative forms of random demand functions. Note also that under perfect competition (1) reduces to $p_t = p(Y_t)$. See Leland (1972, pp. 278-279).

[5] Since the firm produces the good in proportion to the capital endowment, K_t denotes both the firm's stock of capital and the instantaneous production.

[6] The effect of allowing for a constant capital depreciation rate is equivalent to that obtained by raising the discount factor.

[7] See Dixit & Pindyck (1994) for a review of the vast literature on incremental investment under uncertainty.

[8] If no arbitrage opportunities exist and the market is complete, the assumption of risk-neutrality can be relaxed by substituting the drift α with the risk-adjusted one (Cox and Ross, 1976; Harrison and Kreps, 1979).

[9] Having not imposed any restrictions on the rate of investment, dK_t, allows for a singular control.

Using Ito's Lemma, we can expand $dV(K_t, Y_t)$ and rearrange (5) as follows:

$$(1/2)\sigma^2 Y_t^2 V_{KK}(K_t, Y_t) + \alpha Y_t V_Y(K_t, Y_t) - rV(K_t, Y_t) + \Pi(K_t, Y_t) = 0 \tag{6}$$

Standard arguments lead to a solution for (6) taking the following functional form (see Dixit and Pindyck, 1994, chps. 6 and 7):

$$V(K_t, Y_t) = Z(K_t)Y_t^{\beta} + E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt] \tag{7}$$

where $\beta > 1$ is the positive root of the characteristic equation $\Gamma(\beta) = (1/2)\sigma^2\beta(\beta - 1) + \alpha\beta - r = 0$ and $Z(K_t)$ is a constant to be determined. In order to determine $Z(K_t)$ and $Y^*(K_t)$ some suitable boundary conditions are required. These include a value matching condition:

$$V_K(K_t, Y^*(K_t)) = Z_K(K_t)Y^*(K_t)^{\beta} + \left.\frac{\partial E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt]}{\partial K}\right|_{Y=Y^*(K_t)} = c, \tag{8}$$

and a smooth pasting condition:

$$V_{KY}(K_t, Y^*(K_t)) = \beta Z_K(K_t)Y^*(K_t)^{\beta-1} + \left.\frac{\partial^2 E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt]}{\partial K \partial Y}\right|_{Y=Y^*(K_t)} = 0. \tag{9}$$

Substituting (7) into [8 -9] and solving the system yield:

$$Y^*(K_t) = \beta(\left.\frac{\partial E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt]}{\partial K_t}\right|_{Y=Y^*(K_t)} - c)/ \left.\frac{\partial^2 E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt]}{\partial K_t \partial Y_t}\right|_{Y=Y^*(K_t)} \tag{10}$$

$$Z_K(K_t) = (c - \left.\frac{\partial E_0[\int_0^{\infty} \Pi(K_t, Y_t)e^{-rt}dt]}{\partial K_t}\right|_{Y=Y^*(K_t)})/Y^*(K_t)^{\beta} \tag{11}$$

3 Long-run rate of capital accumulation

In this section, we determine, using the results above, the optimal investment policy set by a firm facing two different types of demand shocks, namely isolestic demand with multiplicative shock and linear demand with additive shocks. We will then study the effects of increased uncertainty on the optimal investment policy and the long-run average growth rate of capital accumulation.

3.1 Multiplicative shocks

Assume that the firm faces the following constant elasticity inverse demand function subject to multiplicative shocks:[10]

$$p(K_t, Y_t) = Y_t \cdot K_t^{-\gamma} \tag{12}$$

where $0 < \gamma < 1$. By plugging (12) into [10 - 11], we obtain:

$$Y^*(K_t) = \frac{\beta}{\beta - 1}(r - \alpha)\frac{K_t^{\gamma}}{1 - \gamma}c \tag{13}$$

$$Z_K(K_t) = -\frac{c}{\beta - 1}Y^*(K_t)^{-\beta} \tag{14}$$

As one may easily see, $Y^*(K_t)$ is increasing in K_t. That is, the greater the capital stock, the less likely additional investment becomes in expected terms. Further, (13) says that a marginal increase in the capital stock makes sense only if the expected present value of future additional benefits, $Y^*(K_t)(1-\gamma)K_t^{-\gamma}/(r-\alpha)$, is higher than the unit cost of capital, c, adjusted by the multiple $\beta/(\beta - 1)$ to account for the effect of

[10] Results in this section are standard for readers familiar with the theory of investment under uncertainty. So, we prefer to provide only a brief discussion and address the interested reader to Dixit and Pindyck (1994, chp. 11).

uncertainty on such irreversible decision. Finally, note that, as expected, $Z_K(K_t)$ is negative. This makes economic sense since by $Z_K(K_t)$ we account for the marginal option value implicitly lost by investing in an additional unit of capital. Once determined $Z_K(K_t)$, one may easily find $Z(K_t)$ by integrating (14) on the interval $[K_t, \infty)$. Note, however, that for this integral to converge, we must have $\gamma > 1/\beta$ (or $\beta > 1/\gamma$). That is, returns on capital investment must decrease sufficiently quickly.

Using (13), let's define:

$$\theta_t \equiv \theta(K_t, Y_t) \ = Y_t(1-\gamma)K_t^{-\gamma}/(r-\alpha) \text{ for } t \geq 0, \text{ and } \widehat{\theta} = [\beta/(\beta-1)]c \tag{15}$$

where the process $\{\theta_t\}$ represents the expected discounted net marginal profits of capital and $\widehat{\theta}$ is the marginal cost of a unit of capital adjusted by the option value multiple, $\beta/(\beta-1)$.[11] As one can easily see, (15) describes the dynamic of the optimal investment policy. Note that investment occurs as its profitability θ_t, driven by an increase in Y_t, rises to $\widehat{\theta}$. However, an increase in the stock of capital, i.e. $dK_t > 0$, prevents θ_t from passing above the upper barrier $\widehat{\theta}$.[12]

Although it is not possible to derive a finite rate of capital investment using the reflections at $\widehat{\theta}$ as reference,[13] taking a long run perspective we can determine the average rate of capital accumulation. As first step, we need to check if a steady-state distribution for $\{\theta_t\}$ exists within the range $(0, \widehat{\theta})$. If yes, since for $\theta_t < \widehat{\theta}$ the stock of capital is constant it is always possible to obtain the corresponding marginal probability distribution for K_t. This, in turn, allows us to determine the long-run rate of growth of capital stock.

Defining $\lambda \equiv E(d \ln K_t)/dt$ as a measure of the long-run average rate of capital accumulation, it is easy to show that (see Appendix A.2):

Proposition 1 *Under an isoelastic demand function with multiplicative shocks, the long-run average rate of capital accumulation λ is given by:*

$$\lambda = \begin{cases} [\alpha - (1/2)\sigma^2]/\gamma & \text{for } \alpha > (1/2)\sigma^2 \\ 0 & \text{for } \alpha \leq (1/2)\sigma^2 \end{cases} \tag{16}$$

From the regulated process (15), when θ_t hits the barrier $\widehat{\theta}$ the equality $\gamma \ln K_t = \ln[(1-\gamma)/(r-\alpha)] + \ln Y_t - \ln \widehat{\theta}$ holds. Since $\ln \widehat{\theta}$ does not vary over time, it is evident that the expected evolution of K_t on the boundary is driven only by the expected evolution of $\ln Y_t$. Moreover, since below $\widehat{\theta}$ the stock of capital remains constant, the long-run average growth rate of K_t must be equal to the average growth rate on the boundary.

3.2 Additive shock

Let's now study the case where uncertainty enters additively in the inverse demand function. For the sake of simplicity, let's consider the following functional form:[14]

$$p_t(Q_t, Y_t) = Y_t - K_t^{\eta} \tag{17}$$

where $\eta > 0$ represents the speed at which demand falls as the price increases. Note that for $\eta = 1$, (17) corresponds to the standard linear function.[15] Plugging (17) into [10-11], we obtain:

$$Y^{**}(K_t) \ = \ \frac{\beta}{\beta-1}(r-\alpha)[(\eta+1)\frac{K_t^{\eta}}{r} + c] \tag{18}$$

$$Z_K(K_t) \ = \ -\frac{(\eta+1)\frac{K_t^{\eta}}{r} + c}{\beta-1} Y^{**}(K_t)^{-\beta} \tag{19}$$

[11] As standard in the real option literature, the multiple $\beta/(\beta-1) > 1$ accounts for the presence of uncertainty and irreversibility (Dixit and Pindyck, 1994).

[12] In the technical parlance, θ_t behaves as regulated process with $\widehat{\theta}$ as upper reflecting barrier (Harrison, 1985).

[13] Note in fact that in general we may have long periods of inaction when $\theta_t < \widehat{\theta}$ followed by short periods of rapid bursts of investment whenever θ_t reaches $\widehat{\theta}$. In the first case, no investment occurs and the average rate of capital accumulation is null. In contrast, in the second case, since investment is instantaneous, then the rate of capital accumulation is infinite (see Harrison, 1985).

[14] On the linearity of the demand function see Alperovic and Weksler (1996) studying a class of utility functions which yields demand functions locally linear in prices.

[15] The demand elasticity is given by $(Y_t K_t^{-\eta} - 1)/\eta$. The elasticity is non-increasing in η for $Y_t K_t^{-\eta}(1 + \eta \ln K_t) - 1 \geq 0$.

where it is immediate to note that also in this case we may have a closed form solution for the optimal policy. As above, the barrier, $Y^{**}(K_t)$, is upward-sloping.[16]

For what concerns the impact of uncertainty, the effect is qualitatively similar to the one discussed in the previous section. The same arguments applies also when discussing sign and economic sense of $Z_K(K_t)$. Note that in this case in order to determine $Z(K_t)$ integral convergence requires $\beta > 1 + 1/\eta$.

As shown above, using (18), let define:

$$\xi_t \equiv \xi(K_t, Y_t) = \frac{Y_t}{r-\alpha} - (\eta+1)\frac{\beta}{\beta-1}\frac{K_t^\eta}{r} \text{ for } t \geq 0, \text{ and } \hat{\xi} = [\beta/(\beta-1)]c \tag{20}$$

where the process $\{\xi_t\}$ and $\hat{\xi}$ represent the expected net marginal profits of capital and the marginal cost of a unit of capital adjusted for uncertainty, respectively.

In contrast with the case of multiplicative shock, K_t and Y_t enter additively in (20). This implies that a steady-state distribution for the process $\{\xi_t\}$ can be obtained only by approximating $\xi(K_t, Y_t)$ so that $\xi(K_t, Y_t; \widetilde{K}, \widetilde{Y}) \in (0, \widehat{\xi})$ for any $(\widetilde{K}, \widetilde{Y})$ with $\xi(\widetilde{K}, \widetilde{Y}) \leq \hat{\xi}$. Hence, using $\xi(K_t, Y_t; \widetilde{K}, \widetilde{Y})$, we can determine the marginal distribution of K_t and the long-run average rate of capital accumulation. Denoting by ω the long-run average rate of capital accumulation under a linear demand function and using (20), in the appendix we show that

Proposition 2 *Under a linear demand function with additive shocks, the long-run average rate of capital accumulation ω is given by:*

$$\omega \simeq \begin{cases} \frac{\alpha-\frac{1}{2}\sigma^2}{\eta}\frac{\widetilde{Y}}{\frac{r-\alpha}{r}(\eta+1)\frac{\beta}{\beta-1}\widetilde{K}^\eta} & \textit{for } \alpha > (1/2)\sigma^2 \\ 0 & \textit{for } \alpha \leq (1/2)\sigma^2 \end{cases} \tag{21}$$

Note that when the process ξ_t hits the barrier $\hat{\xi}$, we have $\eta \ln K_t = -\ln\left[\frac{r-\alpha}{r}(\eta+1)\frac{\beta}{\beta-1}\right] + \ln\left[Y_t - (r-\alpha)\hat{\xi}\right]$. Since $Y_t - (r-\alpha)\hat{\xi}$ may also take negative values, it is then not possible to study the dynamic of the capital stock using the expected evolution of Y_t. In appendix A.3, we show how, by linearizing $\ln[Y_t - (r-\alpha)\xi_t]$, one may fix this problem.

By (21), it is straightforward to note that the average long-run rate of growth depends on the initial point $(\widetilde{K}, \widetilde{Y})$. In this respect, by choosing $\widetilde{Y}$ such that $\xi(\widetilde{K}, \widetilde{Y}) = \hat{\xi}$, the expected long-run rate at the boundary becomes:

$$\omega \simeq \begin{cases} \frac{\alpha-\frac{1}{2}\sigma^2}{\eta}(1 + \frac{rc}{\eta+1}\widetilde{K}^{-\eta}) & \text{for } \alpha > (1/2)\sigma^2 \\ 0 & \text{for } \alpha \leq (1/2)\sigma^2 \end{cases} \tag{22}$$

4 Multiplicative vs. additive shocks: a numerical comparison

In order to provide a quantitative measure of the impact of different forms of uncertain demand function on the long-run rate of capital accumulation, we rearrange (22) as follows:

$$\omega = \begin{cases} \lambda\Phi(\widetilde{K}) & \text{for } \alpha > (1/2)\sigma^2 \\ 0 & \text{for } \alpha \leq (1/2)\sigma^2 \end{cases} \tag{23}$$

where $\Phi(\widetilde{K}) = \gamma(1 + \frac{rc}{\eta+1}\widetilde{K}^{-\eta})/\eta$. Note that $\Phi(\widetilde{K}) < 1$ for $1 + \frac{rc}{\eta+1}\widetilde{K}^{-\eta} < \eta/\gamma$.[17] This implies that for any given $\widetilde{K}$, ω will be lower than λ as η increases. Then, in order to better emphasize the difference between ω and λ, let's consider the case where the linearization point $(\widetilde{Y}, \widetilde{K})$ is such that the two demand functions have the same elasticity. That is:

$$(\widetilde{Y}/\widetilde{K}^\eta) = (\eta/\gamma) + 1 \tag{24}$$

[16] Note that with a linear demand function the profit function takes an inverted-U shape. This has a straightforward implication for the investment policy. Note in fact that, due to the rapid decline that marginal capital profitability may have, investment slows down by getting closer to the maximum.

[17] The term $\Phi(\widetilde{K})$ is decreasing and convex in $\widetilde{K}$, i.e. $\Phi'(\widetilde{K}) = -\gamma\frac{rc}{\eta+1}\widetilde{K}^{-(\eta+1)} < 0$ and $\Phi''(\widetilde{K}) = -(\eta+1)\Phi'(\widetilde{K})/\widetilde{K} > 0$.

Substituting (24) into (23) and imposing that $\xi(\widetilde{K},\widetilde{Y}) = \hat{\xi}$ yields:

$$\Omega \equiv \frac{\omega}{\lambda} = \frac{\eta+\gamma}{\eta(\eta+1)\frac{\beta}{\beta-1}\frac{r-\alpha}{r}}, \text{for } \alpha > (1/2)\sigma^2 \tag{25}$$

where Ω is the ratio between the two long-run average rate of capital accumulations. It is easy to see that Ω is decreasing in σ and increasing in γ, and that $\Omega \geq 1$ only for $\eta \leq \bar{\eta} \equiv \frac{-(1/2)\sigma^2\beta+\sqrt{[(1/2)\sigma^2\beta]^2+4r\gamma[r+(1/2)\sigma^2\beta]}}{2r+\sigma^2\beta} < 1$.

In Figure 1 we show some sensitivity analyses of (25). Our choice of parameter values follows numerical examples provided within related field literature (see e.g. Bentolila and Bertola, 1990; Dixit and Pindyck, 1994; Bertola and Caballero, 1994; Bertola,1998): $r = 0.1$; $\alpha = 0.03$; $\sigma \in [0.1.0.25)$ and $\gamma \in [0.6, 1]$.[18] Further, we impose $\eta = 1$ in order to stress the role of a linear demand function.

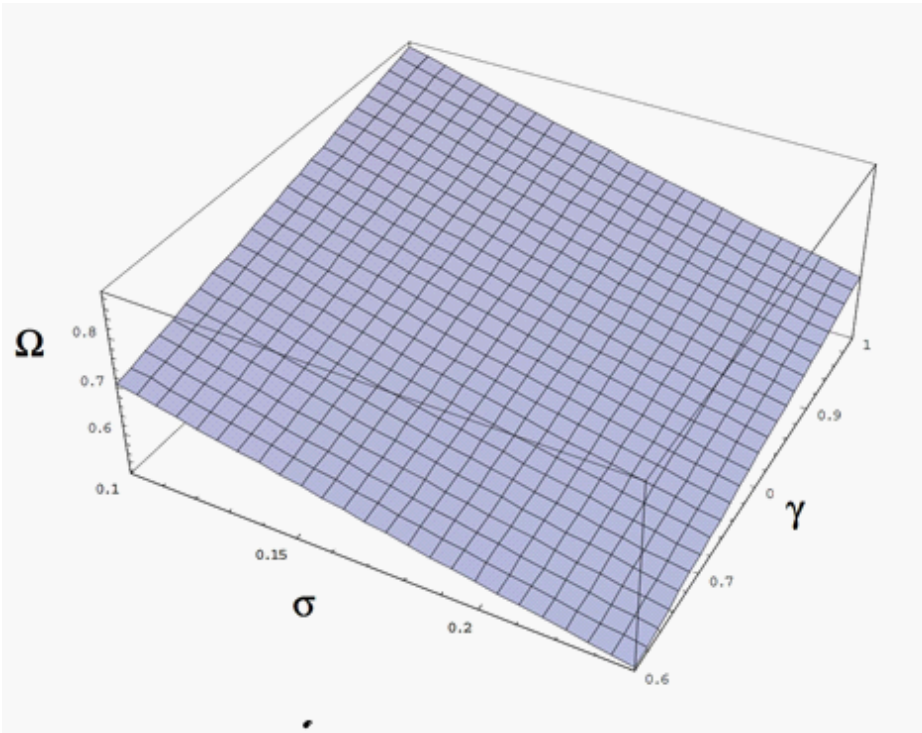

Figure 1: Ω as a function of σ and γ, with $\alpha = 0.03$, $r = 0.1$ and $\eta = 1$.

Note that Ω ranges from 0.52 (for $\sigma = 0.24$ and $\gamma = 0.6$) to 0.88 (for $\sigma = 0.1$ and $\gamma = 1$). Therefore, the difference between the two long-run capital accumulation rates may be significant. In particular, setting $\gamma = 0.6$ and abstracting from the effect of σ we get Ω ranging from 0.71 to 0.52. On the other hand, setting $\sigma = 0.1$ and abstracting from the effect of γ, we get Ω varying from 0.71 to 0.88.

Finally, in Figure 2 we plot Ω as a function of η for $r = 0.1$, $\gamma = 0.6$, $\alpha = 0.03$ and $\sigma = 0.1$. Note that $\Omega \geq 1$ for $\eta < \bar{\eta} = 0.67$. The intuition behind the effect of $\bar{\eta}$ on Ω is that for low level of η, the demand (see

[18] The parameters and the intervals for σ and γ have been taken in order to respect the constraints $\alpha > \frac{1}{2}\sigma^2$ and $\beta > 1/\gamma$ for integrals convergence of (14) and (19) in order to determine $Z(K_t)$.

Eq.(17)) becomes flat. This in turn requires a high level of $\widetilde{K}$ for (24) to hold.

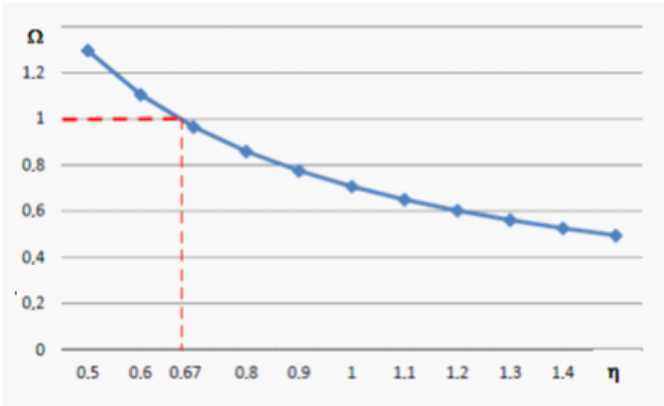

Figure 2: Ω as a function of η, with $\alpha = 0.03$, $r = 0.1$, $\gamma = 0.6$ and $\sigma = 0.1$.

5 Conclusion

This note contributes to study the long-run average growth rate of capital for the case of a linear demand function with a random shock entering additively. This case has not received much attention in the literature of investment under uncertainty since the underlying issue has been generally bypassed by using isoelastic profit functions. We present a technique for the analytical approximation of such rate and show that, by comparing it with the rate determined under a standard log linear profit function, differences may be important.

A Appendix

A.1 Long-run distributions

Let $\{h_t : t \geq 0\}$ be a diffusion process defined in the region $R = (a, b)$ with a and b as left and right boundaries and solving the following differential equation:

$$dh_t = \mu dt + \sigma dW_t \tag{A.1.1}$$

where μ and σ are drift and volatility parameters, respectively, and dW_t is the increment of a Wiener process with $E[dW_t] = 0$ and $E[dW_t^2] = dt$.

Following[19] Harrison (1985, pp. 90-92), the long-run density function for $\{h_t : t \geq 0\}$ is given by the following truncated exponential distribution:[20]

$$f(h) = \begin{cases} (2\mu/\sigma^2)\frac{\exp(2\mu h/\sigma^2)}{\exp(2\mu b/\sigma^2)-\exp(2\mu a/\sigma^2)} & \text{for } \mu \neq 0, \\ 1/(b-a) & \text{for } \mu = 0. \end{cases} \tag{A.1.2}$$

We are interested to the limit case where $a \to -\infty$. In this case, from (A.1.1) a limiting argument gives:

$$f(h) = \begin{cases} (2\mu/\sigma^2)\exp[-2\mu(b-h)/\sigma^2] & \mu > 0, \\ 0 & \mu \leq 0. \end{cases} \tag{A.1.3}$$

Once found its density function, we can now determine the long-run average of h as:

$$\begin{aligned} E[h] &= \int_{-\infty}^{b} hf(h)\,dh = \int_{-\infty}^{b} h(2\mu/\sigma^2)\exp[-2\mu(b-h)/\sigma^2]dh \\ &= (2\mu/\sigma^2)\exp(-2\mu b/\sigma^2)\int_{-\infty}^{b} he^{(2\mu/\sigma^2)h}dh = b - (2\mu/\sigma^2) \end{aligned} \tag{A.1.4}$$

A.2 Proposition 1

By looking at (15), taking the logarithm of θ, we obtain

$$\ln\theta = \ln[(1-\gamma)/(r-\alpha)] + \ln Y - \gamma \ln K \tag{A.2.2}$$

Note that, by using Ito's lemma, the process $\{\ln\theta_t : t \geq 0\}$ evolves according to the diffusion $d\ln\theta = d\ln Y = [(\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW]$ with $\ln\widehat{\theta}$ as upper reflecting barrier. Hence, setting $h = \ln\theta$, the random variable $\ln\theta$ follows a linear Brownian motion with $\mu = \alpha - \frac{1}{2}\sigma^2$ and σ as drift and volatility parameters, respectively, and (A.1.3) as density function.

Taking the expected value on both sides of (A.2.2) and rearranging, we have

$$\begin{aligned} E[\ln K] &= (\ln[(1-\gamma)/(r-\alpha)] + E[\ln Y] - E[h])/\gamma \\ &= [\ln[(1-\gamma)/(r-\alpha)] + Y_0 + [\alpha - (1/2\sigma^2)]t - E[h]]/\gamma \end{aligned} \tag{A.3.3}$$

Since by (A.1.4) $E[h]$ is independent on t, differentiating (A.3.3) with respect to t, we obtain (16) in the text.

A.3 Proposition 2

Let's proceed by taking the logarithm of (20). This yields

$$\begin{aligned} \ln\xi &= \ln\left[\frac{Y}{r-\alpha} - (\eta+1)\frac{\beta}{\beta-1}\frac{K^\eta}{r}\right] \\ &= \ln[1/(r-\alpha)] + \ln[Y - X] \end{aligned} \tag{A.3.1}$$

[19] See Dixit (1993, pp. 58-68) for an heuristic exposition.

[20] Note that hereafter we will drop the time index for notational convenience.

where $X = (\eta+1)\frac{\beta}{\beta-1}\frac{K^\eta}{r}(r-\alpha)$ and $Y > X$. Rewriting $\ln[Y-X]$ as $\ln\left[e^{\ln Y} - e^{\ln X}\right]$ and expanding it by Taylor's theorem around the point $(\widetilde{\ln Y}, \widetilde{\ln X})$ yields:

$$\ln[Y-X] \simeq v_0 + v_1 \ln Y + v_2 \ln X$$

where

$$\begin{aligned} v_0 &= \ln\left[e^{\widetilde{\ln Y}} - e^{\widetilde{\ln X}}\right] - \left[\frac{\widetilde{\ln Y}}{1-e^{\widetilde{\ln X}-\widetilde{\ln Y}}} + \frac{\widetilde{\ln X}}{1-e^{-(\widetilde{\ln X}-\widetilde{\ln Y})}}\right] \\ v_1 &= \frac{1}{1-e^{\widetilde{\ln X}-\widetilde{\ln Y}}},\ v_2 = \frac{1}{1-e^{-(\widetilde{\ln X}-\widetilde{\ln Y})}},\ \frac{v_2}{v_1} = \frac{1-v_1}{v_1} < 0 \end{aligned}$$

By substituting the approximation into (A.3.1) it follows that:

$$\ln\xi \simeq \ln[1/(r-\alpha)] + v_0 + v_1 \ln Y + v_2 \ln X \tag{A.3.2}$$

Now, by Ito's lemma, $\ln\xi$ evolves according to $d\ln\xi = v_1 d\ln Y = v_1\{[\alpha-(1/2\sigma^2)]dt + \sigma dW\}$ with $\ln\hat{\xi}$ as upper reflecting barrier. Setting $h = \ln\xi$, the random variable $\ln\xi$ follows a linear Brownian motion with parameter $\mu = v_1[\alpha-(1/2\sigma^2)] > 0$ and has a long-run distribution with (A.1.3) as density function.

Solving (A.3.2) with respect to $\ln K$ we obtain:

$$\ln K \simeq -(\ln[1/(r-\alpha)] + v_0 + v_1 \ln Y + v_2 \ln\left[(\eta+1)\frac{\beta}{\beta-1}\frac{r-\alpha}{r}\right] - h)/\eta v_2 \tag{A.3.3}$$

Taking the expected value on both sides of (A.3.3) leads to:

$$E[\ln K] \simeq -\frac{\ln[1/(r-\alpha)] + v_0 + v_1\{Y_0 + [\alpha-(1/2\sigma^2)]t\} + v_2 \ln\left[(\eta+1)\frac{\beta}{\beta-1}\frac{r-\alpha}{r}\right] - E[h]}{\eta v_2}$$

Since by (A.1.4) $E(h)$ is independent on t, differentiating with respect to t, we obtain:

$$E[d\ln K]/dt \simeq -[\alpha-(1/2\sigma^2)](v_1/\eta v_2) = [\alpha-(1/2\sigma^2)](e^{\widetilde{\ln X}-\widetilde{\ln Y}}/\eta)$$

By the monotonicity property of the logarithm, $\widetilde{Y}$ must exists such that $\ln\widetilde{Y} = \widetilde{\ln Y}$. Furthermore, by plugging $\widetilde{Y}$ into (18), we can always find a capital stock $\widetilde{K}$ and $\widetilde{X} = (\eta+1)\frac{\beta}{\beta-1}\frac{\widetilde{K}^\eta}{r}(r-\alpha)$ such that a linearization along $(\widetilde{\ln X}, \widetilde{\ln Y})$ is equivalent to a linearization along $(\ln\widetilde{X}, \ln\widetilde{Y})$, where $\widetilde{\ln X} = \ln\widetilde{X}$. This implies that by setting $(\widetilde{Y}, \widetilde{K})$, the long-run average rate of capital accumulation can be written as:

$$\begin{aligned} \omega &= \frac{1}{dt}E[d\ln K] = \{[\alpha-(1/2\sigma^2)]/\eta\}(\widetilde{Y}/\widetilde{X}) = \frac{\{[\alpha-(1/2\sigma^2)]/\eta\}\widetilde{Y}}{(\eta+1)\frac{\beta}{\beta-1}\frac{\widetilde{K}^\eta}{r}(r-\alpha)} \\ &= \{[\alpha-(1/2\sigma^2)]/\eta\}\frac{\frac{\beta}{\beta-1}(r-\alpha)[(\eta+1)\frac{\widetilde{K}^\eta}{r}+c]}{(\eta+1)\frac{\beta}{\beta-1}\frac{\widetilde{K}^\eta}{r}(r-\alpha)} = \{[\alpha-(1/2\sigma^2)]/\eta\}\frac{(\eta+1)\frac{\widetilde{K}^\eta}{r}+c}{(\eta+1)\frac{\widetilde{K}^\eta}{r}} \\ &= \{[\alpha-(1/2\sigma^2)]/\eta\}(1+\frac{rc}{\eta+1}\widetilde{K}^{-\eta}) \end{aligned} \tag{A.3.4}$$

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