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Adjustment Costs and Long Run Spatial Agglomerations¹

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Abstract

We introduce knowledge spillovers as an externality in the production function of competitive firms operating in a finite spatial domain under adjustment costs. Spillovers are spatial as productive knowledge flows more easily among firms located nearby. When knowledge spillovers are not internalized by firms spatial agglomerations may emerge endogenously in a competitive equilibrium, however, they do not emerge at the steady state of the social optimum.

Keywords: Investment theory, adjustment costs, spatial agglomerations
JEL Classification: D21, R3, C61

1. Introduction

The study of adjustment costs in the investment theory of the firm dates back to the 1960s (e.g., Lucas (1967) etc.). A central result obtained by Scheinkman (1978) indicates that in a perfect foresight competitive equilibrium, where firms take the perfect foresight price function as given and face

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convex adjustment costs in net investment, each firm's capital stock converges to a unique steady state which is independent of initial conditions. When firms are identical all firms will converge in the long run to the same stock of capital.

We revisit adjustment costs and the investment theory of the firm by considering a competitive industry operating in a finite spatial domain. Firms can be located at any point in the domain but their production function can be affected by knowledge spillovers stemming from firms located nearby. Knowledge spillovers are regarded as positive intra-industry Marshallian externalities which are bounded in space, the main idea being that innovation and new productive knowledge flows more easily among agents which are located within the same area (e.g. Krugman (1991), Feldman (1999), Breschi and Lissoni (2001)). We introduce these knowledge spillovers as a Lucas/Romer type of externality in the production function. This externality is modeled by a kernel defined over the stock of capital of the firms located in the spatial domain.

We study whether the interplay between adjustment cost in expanding the stock of capital and the knowledge spillovers generated from the expanded capital stock induce endogenous agglomerations and spatial clustering of firms in a competitive industry where profit maximizing firms take the price function and knowledge spillovers as parametric.

Our results suggest that endogenous agglomerations may emerge as a long-run steady state of a perfect foresight rational expectations competitive equilibrium (PF-RECE), where the distribution of capital stocks and outputs across space is not uniform. On the other hand at a social optimum, where a planner endogenizes spatial spillovers, agglomerations do not emerge, and all firms converge to the same stock of capital irrespective of location.

Our contribution is twofold. First we provide a conceptual framework that explains the dynamic endogenous emergence of spatial clustering in a competitive industry. Second, we show how convexity arguments and spectral theory can be used to study PF-RECE problems and social optimum problems in spatiotemporal economies, by properly decomposing the spatial and the temporal behavior. These provide valuable insights regarding the en-

ogenous emergence (or not) of optimal agglomerations at a PF-RECE and the social optimum of competitive industry. The possibility of a potential agglomeration at a PF-RECE is related to the incomplete internalization of the spatial externality by optimizing firms, while the “no agglomerations” result at the social optimum stems from the full internalization of the spatial externality by a social planner and the strict concavity of the production function.

2. Spatial Knowledge Spillovers and Adjustment Costs

We consider an industry consisting of a large number of firms with each firm located at point x of a one-dimensional bounded spatial domain $\mathcal{X} = [-L, L]$.² We assume that \mathcal{X} is discretized, i.e., it is divided into N intervals or cells I_i , $i = 1, \dots, N$, such that $\mathcal{X} = \cup_{i=1}^N I_i$, and to save space we will denote by $\mathcal{N} := \{1, 2, \dots, N\}$ and use the compact notation $i \in \mathcal{N}$. We confine our analysis to a finite dimensional space, because studying a continuum of firms would make the state space infinite dimensional and the mathematical background necessary to study such a problem would exceed space limitations. However, the problem studied here as well as other economic optimization problems are extendable to infinite dimensional state spaces using our methods (e.g. Brock et al. (2012)).

Each firm produces at time $t \in \mathbb{R}_+$ and location $x \in \mathcal{X}$ a single homogenous output $y(t, x)$. To simplify we assume that the output is uniform within each cell or site, i.e. $y(t, x) = y_i(t)$ for every $x \in I_i$, so that the state of the system at time t , is given by a vector $y(t) = (y_1(t), \dots, y_N(t)) \in \mathbb{R}^N$. Local output $y(t, x)$ is produced by two factors of production, local capital stock $k(t, x)$ and accumulated knowledge $K(t, x)$ according to a strictly concave and sufficiently smooth neoclassical production function $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$; $y(t, x) = f(k(t, x), K(t, x))$, with $\frac{\partial^2 f}{\partial k \partial K} > 0$. We also assume that k and K are uniform within each cell, so that $k(t, x)$ is replaced by a vector $k(t) = (k_1(t), \dots, k_N(t)) \in \mathbb{R}^N$, and $K(t, x)$ is replaced by $K(t) = (K_1(t), \dots, K_N(t)) \in \mathbb{R}^N$. Therefore, the production at time t and at cell i is given by $y_i(t) = f(k_i(t), K_i(t))$.

²Most of our results can be extended to general domains of characteristics $\mathcal{X} \subset \mathbb{R}^d$, $d \geq 1$.

Knowledge represents and intra-industry positive externality and is determined, at time t and cell i , by the existing capital stock at nearby cells j . The local capital stock at each cell j , contributes to the total knowledge spillovers at cell i according to a weight function w_{ij} , therefore, the total effect at cell i is

$$K_i(t) = \sum_{j=1}^N w_{ij} k_j(t).$$

We will also use the compact notation $K = Wk$ where $W = (w_{ij})$, $i, j = 1, \dots, N$ is an $\mathbb{R}^{N \times N}$ matrix. If $w_{ij} = 0$ for a pair (i, j) , then cell j does not contribute at all to the total knowledge spillovers at site i . For simplicity we assume that $w_{ij} = \bar{w}(|i - j|)$ for some function \bar{w} , which implies that distance, and not the actual locations, is fundamental in determining non-local effects. Matrix W defines the connectivity of the “knowledge network”. If, for example, $w_{ij} = \delta_{j,i+1} + \delta_{j,i-1} - 2\delta_{j,i}$, where $\delta_{i,j}$ is the Kronecker delta, we have a linear connectivity of the knowledge network, according to which site i interacts only with sites $i + 1$ and $i - 1$. The connectivity of sites 1 and N , in some sense is related with the choice of boundary conditions. If, for example, periodic boundary conditions are imposed so that we consider the network as situated on a circle, then site 1 interacts with site N that is now considered as its neighbor. We wish to emphasize, however, that our analysis is valid for a general choice of network, i.e., for a general choice of matrix W .

An important class of networks are those that satisfy the condition $\sum_j w_{ij} = \bar{w}$, independent of the choice of i . A particular example for such a coupling is the matrix $w_{ij} = \delta_{j,i+1} + \delta_{j,i-1} - 2\delta_{j,i}$ which satisfies this condition for $\bar{w} = 0$. We will call such couplings diffusive type couplings. It means that if the stock of capital is uniform across all sites and equal to k then every site i is going to experience an externality equal to $\bar{w}k$. The adoption of this condition on the network will allow us to establish some important and general results concerning the possibility or not of emergence of spatial agglomerations in the economy.

Knowledge externality $K_i(t)$ will have different interpretations in different contexts. If $K_i(t)$ represents a type of knowledge which is produced

proportionately to capital usage, it is natural to assume that the kernel \bar{w} considered as a function of $\zeta = i - j$ is single peaked and bell-shaped, with a maximum at $\zeta = 0$, and of sufficiently fast decay to 0 for sufficiently large $|\zeta|$. If $K_i(t)$ reflects aggregate benefits of knowledge produced at (t, i) for producers at (t, i) and damages to production at (t, i) from usage of capital at (t, j) , then non-monotonic shapes of \bar{w} with, for example, a single peak at $\zeta = 0$ and two local minima located symmetrically around $\zeta = 0$, with negative values indicating damages to production at i from usage of capital at j , are plausible. This production function could be considered as a spatial version of a neoclassical production function with Romer/Lucas externalities modelled by geographical spillovers given by a Krugman (see e.g., Krugman (1996)), or Chincarini and Asherie type specification (see e.g. Chincarini and Asherie (2008)).

The temporal rate of change of capital stock is given by the derivative with respect to time, k' , of the vector valued function $k : \mathbb{R}_+ \rightarrow \mathbb{R}^N$. The firm faces a cost of changing the capital stock, which depends on the value of the function k' . This adjustment cost at time t and cell i is expressed by a quadratic adjustment function $C_i(t) = \frac{\alpha}{2}(k'_i(t))^2$, $\alpha > 0$. Capital stock depreciates at the same rate η in all sites.

The output of the firms is sold at a market price determined by a demand function $D : \mathbb{R} \rightarrow \mathbb{R}_+$.

$$p(t) = D(Q) = D(Q(k, K)), \quad D > 0, D' < 0 \tag{1}$$

$$Q := Q(k, K) = \sum_{i=1}^N f(k_i(t), K_i(t)). \tag{2}$$

The k and K dependence is stated explicitly to emphasize that D can be understood as a functional $D : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$; given k , we obtain $K = Wk$, and calculate the total output Q that determines the price p . Assuming perfect capital markets and that the unit price of capital is q independent of time, the objective of a firm located at $i \in \mathcal{N}$ is to maximize the present value of profits by considering knowledge spillovers as exogenous $K_i = K_i^e$.

The firm's problem can be written as:

$$\max_{k'_i} \int_0^\infty e^{-rt} \left[p(t) f(k_i, K_i^e) - \frac{\alpha}{2} (k'_i)^2 - q(k'_i + \eta k_i) \right] dt \quad (3)$$

$$k_i(0) = k_{i0}, k_i(t) \geq 0, \quad i \in \mathcal{N}. \quad (4)$$

Given this set up we define the industry equilibrium and the social optimum and explore conditions that could generate endogenous spatial clustering of firms.

3. Industry Equilibrium and Social Optimum

Following (Lucas and Prescott (1971), Brock (1974), Brock and Scheinkman (1977)) we define a PF-RECE as the price function $p(t)$ given by (1) where $k_i(t)$ solves (3) for all $i \in \mathcal{N}$ with optimality conditions evaluated at $K^e = Wk$. If the price path $p(t)$ is predicted by the competitive firms, this path will result in an aggregate output Q over the whole spatial domain such that the market is cleared at each t by $p(t)$.

The long-run properties of the industry equilibrium can be obtained by exploiting the technique of maximizing consumer surplus (Lucas and Prescott (1971), Brock (1974), Brock and Scheinkman (1977)). Two optimization problems leading to different concepts of equilibria can be defined in this context:

- (A) The problem of maximizing consumer surplus when firms are regarding knowledge spillovers as exogenous, that is when they do not internalize the spatial externality and they set $K_i(t) = K^e$, defined as:

$$\max_{k'} \int_0^\infty e^{-rt} \left\{ S(k, K^e) - \sum_{i=1}^N \left[\frac{\alpha}{2} (k'_i)^2 - q(k'_i + \eta k_i) \right] \right\} dt \quad (5)$$

$$S(k, K) = \int_0^{Q(k, K)} D(u) du \quad (6)$$

Thus the firm treats K^e as parametric, but the actions of all firms generate the "actual" value of the realized knowledge externality which is Wk and which determines the equilibrium outcome. The solution to this problem for $K^e = Wk$ determines the PF-RECE.

(B) The problem of maximizing consumer surplus when a social planner fully internalizes the spatial externality. This means

$$\max_{k'} \int_0^\infty e^{-rt} \left\{ S(k, Wk) - \sum_{i=1}^N \left[\frac{\alpha}{2} (k'_i)^2 - q(k'_i + \eta k_i) \right] \right\} dt \quad (7)$$

The solution to this problem determines the social optimum.

The Euler equations for these two problem can be obtained in a straightforward manner, using e.g. the Pontryagin maximum principle. A straightforward analysis leads to expressing the Euler equations for the PF-RECE problem (5) as

$$k''_i - rk'_i + \frac{1}{\alpha} \left[\frac{\partial}{\partial k_i} S(k, K^e) \Big|_{K^e=Wk} - q(r + \eta) \right] = 0, \quad i \in \mathcal{N}, \quad (8)$$

where the notation $\frac{\partial}{\partial k_i} S(k, K^e) \Big|_{K^e=Wk}$ means that we first take the gradient of $S(k, K^e)$ with respect to k , treating K^e as fixed, and then substitute $K^e = Wk$ in the resulting function to determine the PF-RECE. For the social optimum, problem (7), the Euler equation is:

$$k''_i - rk'_i + \frac{1}{\alpha} \left[\frac{\partial}{\partial k_i} S(k, Wk) - q(r + \eta) \right] = 0, \quad i \in \mathcal{N}. \quad (9)$$

This leads to the following definition:

Definition 1 A solution $k : \mathbb{R}_+ \rightarrow \mathbb{R}^N$ of (8), with $K^e = Wk$ is called a PF-RECE. while a solution of (9) is called a social optimum.

Note that in the social optimum $\frac{\partial}{\partial k_i} S(k, Wk)$ are the components of the true gradient of function S , treated as a function of k only, i.e., the true gradient of the function $S(k, Wk)$. This is in contrast to what happens for the PF-RECE where $\frac{\partial}{\partial k_i} S(k, K^e) \Big|_{K^e=Wk}$, no longer correspond to the components of a “true” gradient of a function. This remark plays a very important role to the qualitative long term behavior of the two systems and leads to important differences between them.

4. Long term behavior of the social optimum: A global result

We provide a global result concerning the spatial structure, i.e. the possibility of agglomerations, as a long run outcome at the social optimum.

Proposition 2 *If the system of equations*

$$\frac{\partial}{\partial k_i} S(k, Wk) - q(r + \eta) = 0, \quad i \in \mathcal{N}, \quad (10)$$

admits the spatially uniform $k_1 = \dots = k_N = \bar{k}$ solution, then no spatial patterns are admissible in the long run equilibrium for the social optimum.

Proof: Define the functions $\bar{S} : \mathbb{R}^N \rightarrow \mathbb{R}$, by $\bar{S}(k) := S(k, Wk) - q(r + \eta)k$ and $\hat{S} : \mathbb{R} \rightarrow \mathbb{R}$ by $\hat{S}(x) = \int_0^x D(u)du$. The function $\hat{S}(x)$ is strictly concave as the integral of a strictly decreasing function, and by the properties of the production function the function $S(k, Wk)$ a strictly concave function of k . Therefore, function $\bar{S}(k)$ is strictly concave. The Euler equation can be written as

$$k'' - rk' = -\nabla \bar{S},$$

and by the convexity of $-\bar{S}$, the operator $-\nabla \bar{S}$ is a monotone operator on \mathbb{R}^N . By Theorem 3.3 of Rouhani and Khatibzadeh (2009) any bounded solution of these systems converges to the steady state which is a solution of (10).³ The solution of this equation is recognized as the minimum of the function $-\bar{S}$ respectively, which is unique by strict convexity. Therefore, the result follows. **QED**

Finally we provide below conditions under which no agglomerations are possible at the social optimum.

Assumption 3 *The coupling is of diffusive type, i.e. $\sum_j w_{ij} = \bar{w}$ for any $i \in \mathcal{N}$, and the production function is homogeneous of degree γ .*

Proposition 4 *Let Assumption 3 hold. If the scalar algebraic equation*

$$\gamma N^{\frac{1-\gamma}{\gamma}} \rho^{\frac{1}{\gamma}} D(s) s^{\frac{\gamma-1}{\gamma}} - q(r + \eta) = 0, \quad \rho := f(1, \bar{w}) \quad (11)$$

³This approach is readily extendable to infinite dimensional state spaces.

admits a solution $s^* \in \mathbb{R}_+$, then no agglomeration patterns may appear in the long run equilibrium for the social optimum and the industry relaxes to a spatially homogeneous state $k_1 = \dots = k_N = k^* = \left(\frac{s^*}{N\rho}\right)^{\frac{1}{\gamma}}$.

Proof: The steady state is given by the solution of the system of equations

$$D(Q(k, Wk)) \left(f_k(k_i, \sum_j w_{ij}k_j) + \sum_r w_{ri}f_K(k_i, \sum_j w_{rj}k_j) \right) - q(r + \eta) = 0, \quad i \in \mathcal{N},$$

which for a spatially uniform solution $k_1 = \dots = k_N = k^*$ and using Assumption 3 reduces to single algebraic equation, which is equivalent to (11), in terms of the variable $s = N\rho(k^*)^\gamma$. Then using Proposition 2 we obtain the stated result. **QED**

5. Agglomeration patterns in the perfect foresight rational expectations equilibrium

The situation is different for the PF-RECE, where the term $\frac{\partial}{\partial k_i} S(k, K^e) \Big|_{K^e=Wk}$ is no longer a gradient so that in general we may not have a result similar to Proposition 2. Therefore, spatial agglomerations may emerge through the perturbation of a spatially homogeneous steady state, in a fashion which is similar (but different in mechanism) to the celebrated Turing instability. The next proposition presents such a case.

Proposition 5 *Let Assumption 3 hold, and define the real numbers $\rho_k := f_k(1, \bar{w})$, $\rho_K := f_K(1, \bar{w})$, $\rho_{kK} := f_{kK}(1, \bar{w})$.*

1. *If the scalar algebraic equation $\left(\frac{1}{N\rho}\right)^{\frac{\gamma-1}{\gamma}} \rho_k D(s) s^{\frac{\gamma-1}{\gamma}} - q(r + \eta) = 0$ admits a solution $s_* \in \mathbb{R}_+$, then a spatially homogeneous steady state $k_* = \left(\frac{s_*}{N\rho}\right)^{\frac{1}{\gamma}}$ exists.*

2. *Suppose 1 is true and define the matrix $T := C_1 I + C_2 W + C_3 \mathbf{1}$, where I is the $N \times N$ identity matrix, $\mathbf{1}$ is an $N \times N$ matrix consisting of 1's and*

$$C_1 := \frac{1}{\alpha} k_*^{\gamma-2} D(s_*) \rho_{kk}, C_2 := \frac{1}{\alpha} k_*^{\gamma-2} D(s_*) \rho_{kK}, C_3 := \frac{1}{\alpha} k_*^{2(\gamma-1)} D'(s_*) \rho_k (\rho_k + \bar{w} \rho_K),$$

(i) *If the matrix T has eigenvalues greater than $\frac{r^2}{4}$, then pattern formation (agglomerations) may appear, whereas (ii) if T has positive eigenvalues but*

less than $\frac{r^2}{4}$, then a temporally oscillating spatial agglomeration may appear around the spatially homogeneous steady state k_* .

Proof: 1. The steady state will be a solution of

$$D(Q(k, Wk)) \left(f_k(k_i, \sum_j w_{ij} k_j) \right) - q(r + \eta) = 0, \quad i \in \mathcal{N},$$

and the proof proceeds using similar arguments as in Proposition 4.

2. We now look for the evolution of a spatially non homogeneous perturbation of this homogeneous steady state. Consider a solution of (8) of the form $k_i = k_* + \epsilon p_i$, $i \in \mathcal{N}$, where ϵ is a small parameter. We substitute into (8) and linearize with respect to ϵ . After some tedious algebra, and keeping in mind the properties of the production function we obtain the linearized system,

$$p'' - rp' + Tp = 0. \quad (12)$$

The matrix T is symmetric, so there exists an orthonormal basis of \mathbb{R}^N consisting of the eigenvectors of T . Projecting (12) along the eigenvectors, the general solution of (12) can be expressed as $p(t) = \sum_{\ell=1}^N q_\ell(t) \phi_\ell$ where $q_\ell'' - rq_\ell' + \lambda_\ell q_\ell = 0$, $\ell \in \mathcal{N}$, and now the system is decoupled, with its behavior given in terms of the characteristic roots $v_\ell^\pm = \frac{1}{2}(r \pm \sqrt{r^2 - 4\lambda_\ell})$, $\ell \in \mathcal{N}$ which lead to 3 possibilities: (A) $\frac{r^2}{4} < \lambda_\ell$, so that $v_\ell^\pm = \frac{r}{2} \pm i\sigma$, i.e., a pair of complex conjugate roots. (oscillatory behavior compatible with the transversality condition - Hopf type behavior). (B) $0 < \lambda_\ell < \frac{r^2}{4}$, so that $v_\ell^- < \frac{r}{2} < v_\ell^+$, i.e., a pair of real roots, one larger (thus rejected by transversality) and one smaller than $\frac{r}{2}$ (leading to instability as long as it is positive). (C) $\lambda_\ell < 0$, so that $v_\ell^- < 0 < \frac{r}{2} < v_\ell^+$, i.e., a pair of real roots, one negative (thus stable) and one positive larger than $\frac{r}{2}$ (thus rejected by transversality). Thus case B could lead to pattern formation. **QED**

Note that: In general $k^* \neq k_*$, i.e. the steady state of the social optimum problem does not coincide with the steady state of the RE-PFCE; case (i) is reminiscent of a Turing instability with the major difference that is related to a controlled system, which implies that all behavior has to be compatible with the transversality condition; case (ii) is in turn related to a Hopf

type bifurcation. Furthermore, the conditions for pattern formation in the linearized problem are related to the spectrum of the symmetric matrix T , which is easily computed for concrete applications, numerically. The concavity of the production function and the monotonicity of the demand function, provide important information on the signs of the constants C_1, C_2, C_3 and thus allow us to obtain general information concerning the position of the spectrum of the matrix T .

Finally, and most important for the economics of the industry, in the PF-RECE, we do not expect in general an analogue of Proposition 2, since as observed in the beginning of this section the term $\left. \frac{\partial}{\partial k_i} S(k, K^e) \right|_{K^e=Wk}$ is no longer a gradient. The local behavior described for the linearized system around the homogenous steady state \bar{k} , by Proposition 5 suggests that it is possible for some of the unstable modes leading to spatial patterns for the linearized system to persist, leading thus the PF-RECE to long-run stable agglomerations. It is interesting to note that this is in contrast to the socially optimum, where agglomerations and clustering in the long run are definitely ruled out by Proposition 2. In terms of economics this means that diminishing returns in both the stock of capital and knowledge spillovers, expressed by $f_{kk}(k, K), f_{KK}(k, K), K = Wk$ respectively, eradicate any spatial patterns when knowledge spillovers are internalized at the firm level. When, however, knowledge spillovers are not internalized then interactions of the complementarity between the stock of capital and the knowledge spillover expressed by $f_{kK}(k, K)$, with the diminishing returns in the stock of capital, may induce the emergence of spatial agglomeration which could become persistent.

6. Concluding Remarks

We revisit adjustment costs and the investment theory of the firm in a spatial context where knowledge spillovers, which are regarded as a positive externality in the production function, are determined by spatial proximity of firms. We show that spatial agglomerations may emerge endogenously in a PF-RECE where firms do not internalize spatial knowledge spillovers, however, they do not emerge at the social optimum when knowledge spillovers are internalized. Our result suggest therefore that agglomerations are possible as a long run equilibrium outcome in a competitive industry with spatial

knowledge spillovers without the presence of increasing returns.

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