# THE STRUCTURE OF THE COST OF CAPITAL UNDER UNCERTAINTY

Avraham Beja[*]

## 1  Introduction

In the analysis of models of competitive markets under uncertainty, different approaches can be distinguished. One approach, typically dealt with in welfare economics, is the specification of environmental conditions that are sufficient for the existence of equilibrium prices, with all the corollary implications for the efficiency of the competitive system. Prominent examples of this approach are the works of Arrow [2], Debreu [3] and Radner [12]. Another approach, typical to the analysis of capital markets, is concerned more with the structure of equilibrium prices, rather than their existence. Under this approach, basic interest lies in the implications of various given assumptions about the preferences of individuals (e.g. risk aversion) on the relationship between various properties of capital assets and their equilibrium prices, which are initially assumed to exist. Such studies extend from the classical investigations, such as Hicks [6] and Lutz [10], on the term to maturity of "riskless" capital assets, to recent ones concerned primarily with the issue of risk, such as Sharpe [13], Lintner [9], Hirshleifer [7,8] and others.[1]

The present study represents an attempt at a slightly different approach. We postulate the existence of equilibrium prices for capital assets under uncertainty, and then proceed to analyze the properties implicit in their definition. It is shown that some results can be derived without recourse to the way individuals make decisions, their detailed preferences or their subjective assessments of probabilities.

[*]The author appreciates the must useful comments of two referees.

[1]For example, see also Diamond [4], Green [5], and Myers [11] - all using the same Arrow-Debreu framework of uncertainty used here.

## 2  Definition of Prospects and Their Worth

We shall use the "state-space description of uncertainty"[2] of Arrow [2] and Debreu ([3], ch. 7). The set of (future) time points of interest is denoted by $T$ and for simplicity of exposition is assumed finite. The state space $S = \{s\}$ is partitioned for any $t \in T$ into non empty sets $e_t^i$ $i = 1...K_t$, called "events at $t$" ($K_0 = 1$). If $\tau > t$ then the partition corresponding to $\tau$ is finer than the partition for $t$. Contracts for streams of cash flows, to be called here "prospects" are defined by a (possibly explicit but usually implicit) specification of the set of (dollar) values $z\left(e_t^i\right)$ for all $t \in T$, $i = 1...K_t$.

The set $Z$ of all conceivable prospects may be considered as a real vector space by the usual rules of addition and scalar multiplication, namely $z = az^1 + bz^2$ if and only if $z\left(e_t^i\right) = az^1\left(e_t^i\right) + bz^2\left(e_t^i\right)$ for all $t \in T, i = 1...K_t$. The dimension of $Z$ is clearly $K = \sum\limits_{t \in T} K_t$.

Our basic objective is to analyze the structure of equilibrium prices for prospects, as implied by essentially only equilibrium properties. Thus we are not concerned here with conditions for the existence of these prices, nor do we assume that prospects, as defined above, are the actual domain of individual preferences. We therefore choose as a starting point the assumption that equilibrium prices exist for all conceivable prospects. Formally, let the functional $q$ be defined on $Z$ so that, for any $z \in Z, q(z)$ denotes the "worth" of prospect $z$. Worth is defined as the equilibrium price when the market is completely informed about the prospect specification of all contracts. The functional $q$ will be called the "cost of capital".[3]

It is well known that equilibrium properties require that the functional $q$ be linear. Any linear functional defined on $Z$ can be represented by a set of values

$$\left\{q\left(e_t^i\right), t \in T, i = 1...K_t\right\},$$

to be called here "elementary prices", so that for any $z \in Z$

$$q(z) = \sum_{t \in T} \sum_{i=1}^{K_t} z\left(e_t^i\right) \ q\left(e_t^i\right); \qquad\qquad ...(2.1)$$

$q\left(e_t^i\right)$ is to be interpreted as "the equilibrium price at time 0 of the right to \$1 to be delivered at time $t$ if and only if $e_t^i$ obtains".

---

[2]To emphasize that we do not assume the state-space framework to be the basis for investors' preferences or decisions, we deliberately avoid the term "state preference model " frequently used in the literature.

[3]We shall show in section III how this terminology fits with the conventional concept of the "cost of capital" as a number, or percentage, indicating a "rate of interest".

# 3    The Structure of the Cost of Capital

The determination of the worth of any given prospect is in principle a matter for observation—one simply has to go to the market and find out. It may still be of interest to evaluate the worth of a prospect $z$ by inference from the worth of other prospects. Conceptually, this problem is also trivial: one has to evaluate the elementary prices $\left\{q\left(e_t^i\right)\right\}$(by inference) from market data, determine the detailed prospect representation $\left\{z\left(e_t^i\right)\right\}$, and then compute $q\left(z\right)$ as in equation (2.1). Computationally, the dimension $K$ of the space $Z$ is a serious problem. It is thus a question of interest whether the worth of a prospect can be evaluated through the use of a smaller amount of data. Essentially, we want to search for "sufficient statistics" for the evaluation of worth. The analysis of the worth functional will provide interesting insight in this direction. We shall first concentrate on prospects involving only a single (uncertain) cash flow. Such prospects will be denoted by $z_t$, the notation implying that

$$z_t\left(e_\tau^i\right) = 0 \text{ for all } \tau \neq t, i = 1...K_\tau. \qquad ...(3.0.1)$$

Note that any prospect is the sum of prospects involving only a single cash flow, and thus no genreality is lost.

## 3.1    The Riskless Interest Rate

Consider a contract for the delivery of \$1 at time $t$ with certainty. The prospect representation of this contract is as follows:

$$\begin{aligned} z_t\left(e_\tau^i\right) &= 1 \quad \text{for} \quad \tau = t, i = 1...K_\tau, \\ &= 0 \quad \text{for} \quad \tau \neq t, i = 1...K_\tau. \end{aligned}$$

Denoting the worth of this special contract by $\rho_0\left(t\right)$, we have

$$\rho_0\left(t\right) = \sum_{i=1}^{K_t} q\left(e_t^i\right). \qquad ...(3.1.1)$$

If we denote by $i_{0t}$ the interest rate at time 0 for riskless loans with maturity at $t$, we have by definition

$$\rho_0\left(t\right) = \frac{1}{\left(1 + i_{0t}\right)^t}. \qquad ...(3.1.2)$$

## 3.2    The Certainty Equivalent

Consider any single cash flow prospect $z_t$. Equation (2.1) may be rewritten as

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{i=1}^{K_t} \frac{q\left(e_t^i\right)}{\rho_0\left(t\right)} z_t\left(e_t^i\right). \qquad ...(3.2.1)$$

Let

$$\frac{q\left(e_t^i\right)}{\rho_0\left(t\right)} = \nu\left(e_t^i\right),$$

and thus

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{i=1}^{K_t} \nu\left(e_t^i\right) \ z_t\left(e_t^i\right); \qquad \text{...(3.2.2)}$$

$\nu\left(e_t^i\right)$ is the ratio of the present worth of \$1 at time $t$ contingent on event $e_t^i$ to the present worth of \$1 at time $t$ with certainty. This may be safely assumed non-negative, and together with (3.1.1) we have for all $t$

$$\nu\left(e_t^i\right) \geqq 0, \quad i = 1...K_t,$$

$$\sum_{i=1}^{K_t} \nu\left(e_t^i\right) = 1.$$

It will be convenient to let also

$$\sum_{i=1}^{K_t} \nu\left(e_t^i\right) \ z_t\left(e_t^i\right) = C\left(z_t\right), \qquad \text{...(3.2.3)}$$

so that

$$q\left(z_t\right) = \rho_0\left(t\right) \ C\left(z_t\right). \qquad \text{...(3.2.4)}$$

$C\left(z_t\right)$ is thus a weighted average of the values $z_t\left(e_t^i\right)$ that the prospect $z_t$ can take, where the weights $\nu\left(e_t^i\right)$ are determined by the market (since they are completely defined by $q$). The worth of any cash flow $z_t$ is equal by (3.2.4) to $C\left(z_t\right)$ discounted at the riskless discount factor, or equivalently to the worth of a riskless cash flow $C\left(z_t\right)$ to be delivered at the same time. It is thus natural to call $C\left(z_t\right)$ *the certainty equivalent* of $z_t$, and consider the worth of $z_t$ as composed of a pure time factor $\rho_0\left(t\right)$ and a certainty equivalent embodying all risks involved in $z_t$.

## 3.3    An Exogenous Probability Measure

Individuals will generally differ in their assessments of the probabilities of the events $e_t^i$ that may obtain at time $t$. It is therefore meaningless to consider a "market probability measure" on the state space $S$.

One may be tempted at this point to coniser the weights $\nu\left(e_t^i\right)$ as surrogate market probabilities. This fails, however, because the markets' weights $\nu\left(e_t^i\right)$ need not (and generally will not) satisfy a basic requirement of a probability measure, namely that for any $t$ and $\tau$ $(\tau > t)$

$$\text{if} \quad \bigcup_{i=m}^{n} e_\tau^i = e_t^j \quad \text{then} \quad \sum_{i=m}^{n} \nu\left(e_\tau^i\right) = \nu\left(e_t^j\right). \qquad \text{...(3.3.1)}$$

We shall demonstrate presently why it is indeed also implausible to assume that the weights $\nu\left(e_t^j\right)$ will somehow satisfy equation (3.3.1).

From an intuitive viewpoint, it is very convenient to analyze uncertain cash flows using the notions of "expected value", "variance" etc., that are associated with some probability measure. We shall therefore now introduce an exogenous measure $\pi$ on the state space $S$, that may be conveniently considered as given objective probabilities. Needless to say, the subjective probability assessments of agents need not agree with $\pi$ (most of our arguments are not substantially changed if $\pi$ is interpreted not as objective probabilities, but as the sujbective probabilities of the analyst or of the decision-maker). We want to analyze the structure of the worth of prospects with respect to the measure $\pi$. Formally, let $\pi$ be defined on $S$ so that

$$\pi\left(s\right) \geqq 0, \quad \sum_{s \in S} \pi\left(s\right) = 1,$$

and

$$\pi\left(e_t^i\right) = \sum_{s \in e_t^i} \pi\left(s\right).$$

We shall restrict our attention to events with positive probability, so that

$$\pi\left(e_t^i\right) > 0,$$

$$\sum_{i=1}^{K_t} \pi\left(e_t^i\right) = 1 \quad \text{for all } t \in T.$$

We shall also assume that for all these non-null evetns $\nu\left(e_t^i\right) > 0$. This is essentially a requirement that for any non-null event there should be at least some rational agent that also subjectibely assigns to it a positive probability. If $\pi$ is the subjective assessment of an active decision maker, this requirement holds automatically.

The definition of $\pi$ implies well defined probability distributions for all cash flows. Note, however that the existence of probability distributions and net worth values for all cash flows does not imply a well defined mapping from probability distributions at given time points to the net present worth of these distributions. In fact, two cash flows with identical marginal probability distributions may have widely differing net worth values.[4] On the basis of the

---

[4]For example, consider two unions $S^1$ and $S^2$ of events at $t$ such that

$$\sum_{e_t^i \in S^1} \pi\left(e_t^i\right) = \sum_{e_t^i \in S^2} \pi\left(e_t^i\right),$$

and

$$\sum_{e_t^i \in S^1} \nu\left(e_t^i\right) \neq \sum_{e_t^i \in S^2} \nu\left(e_t^i\right).,$$

Clearly this is possible unless $\nu$ is indentically equal to $\pi$ - a case which will be shown later to be implausible and uninteresting. Now consider two uncertain cash flows at $t$: one takes

discussion to this point it is not clear whether the joint distribution of a concrete cash flow with all other available cash flows is sufficient to determine its worth (in section V we shall see that under some additional restrictions defined by the notion of a "risk averse market" this joint distribution is a plausible starting point for empirical analysis of worth).

In the analysis of the worth of cash flows relative to a given measure $\pi$, we expect a return contingent on a likely event to be worth more than a similar return contingent on an unlikely event. The valuation of contingent returns net of the time element and of probability will prove very important in our analysis. Formally, define random variables $\{\beta_t,\ t \in T\}$ by

$$\beta_t\left(e_t^i\right) = \frac{\nu\left(e_t^i\right)}{\pi\left(e_t^i\right)}, \qquad \qquad ...(3.3.2)$$

so that $\beta_t\left(e_t^i\right)$ is the ratio of the present worth of a cash flow at time $t$ contingent $e_t^i$ to the present worth of a riskless cash flow at the same time and with the same expected value.[5]

We can use $\beta$ to demonstrate that the market weights $\nu\left(e_t^i\right)$ cannot serve as surrogate probabilities, and that the violation of (3.3.1) is not inconsistent with the rational behaviour. Consider, for example, an extremely idealized situation where all investors are the same, their assessment of the probabilities is $\pi$ and they maximize the expected value of an additive utility. Then by the classical Lagrangean analysis, for any $i$ and $j$ we must have

$$\frac{q\left(e_t^i\right)}{q\left(e_t^j\right)} = \frac{\pi\left(e_t^i\right)u'\left(\omega\left(e_t^j\right)\right)}{\pi\left(e_t^j\right)u'\left(\omega\left(e_t^i\right)\right)}, \qquad \qquad ...(3.3.3)$$

where $u$ is the utility of wealth (assumed the only argument of this function and $\omega\left(e_t^i\right)$ the investors' wealth in the event $e_t^i$. But

$$\frac{q\left(e_t^i\right)}{q\left(e_t^j\right)} = \frac{\nu\left(e_t^i\right)}{\nu\left(e_t^j\right)},$$

and hence

$$\frac{\beta\left(e_t^i\right)}{\beta\left(e_t^j\right)} = \frac{u'\left(\omega\left(e_t^i\right)\right)}{u'\left(\omega\left(e_t^j\right)\right)}. \qquad \qquad ...(3.3.4)$$

---

the value 1 on $S^1$ and 0 elsewhere, and the other takes the value 1 on $S^2$ and 0 elsewhere. These two cash flows have identical probability distributions and different net worth values.

[5]To see this, consider a prospect $z_t$ of $x$ dollars at $t$ contingent on $e_t^i$. Clearly

$$q\left(z_t\right) = xq\left(e_t^i\right) = x\nu\left(e_t^i\right)\rho_0\left(t\right)$$

and the expected value of $z_t$ is $x\pi\left(e_t^i\right)$. The present worth of the amount $x\pi\left(e_t^i\right)$ at $t$ with certainty is naturally $x\pi\left(e_t^i\right)\rho_0\left(t\right)$, and the conclusion on $\beta$ follows.

In this idealized situation any meaningful "surrogate" probabilities must actually equal $\pi$. But with non linear utility we must have $\beta\left(e_t^i\right) \neq \beta\left(e_t^j\right)$ whenever $\omega\left(e_t^i\right) \neq \omega\left(e_t^j\right)$, so that $\beta$ is not identically 1, $\nu$ cannot equal $\pi$, and cannot therefore be a meaningful surrogate. To see that $\nu$ cannot be expected to be "probabilities" in the sense of (3.3.1), suppose that under the same model $\omega\left(e_t^i\right)$ can take only one of the values $A$ (e.g. affluence) or $R$ (e.g. recession). Let $\beta\left(e_t^i\right) = \beta_A$ if $\omega\left(e_t^i\right) = A$ and $\beta\left(e_t^i\right) = \beta_R$ if $\omega\left(e_t^i\right) = R$. Recall that $\tau > t$, and let $S_A$ denote the (non null) union of all states $e_\tau^i$ such that $e_\tau^i \subset e_t^j$ and $\omega_\tau^i = A$, and similarly for $S_R$. Now rewrite (3.3.1)

$$\text{if} \quad \bigcup_{i=m}^n e_\tau^i = e_t^j \quad \text{then} \quad \sum_{i=m}^n \nu\left(e_t^i\right) = \nu\left(e_t^j\right), \qquad \qquad ...(3.3.1)$$

and note that

$$\nu\left(e_t^j\right) = \pi\left(e_t^j\right)\beta\left(e_t^j\right)$$

and

$$\sum_{i=m}^n \nu\left(e_\tau^i\right) = \pi\left(S_A\right)\beta_A + \pi\left(S_R\right)\beta_R.$$

By definition $\pi\left(S_A\right)\beta_A + \pi\left(S_R\right) = \pi\left(e_t^j\right)$, and since both are assumed strictly positive, $\beta\left(e_t^j\right)$ must lie strictly between $\beta_A$ and $\beta_R$ to satisfy (3.3.1). But by assumption $\beta\left(e_t^j\right)$ is either $\beta_A$ or $\beta_R$, so that (in this case) the requirements are not met.

## 3.4 Evaluating Prospects by their Probabilistic Properties

To see how the worth is expressed in terms of the probabilities, rewrite equation (3.2.2) as

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{i=1}^{K_t} \pi\left(e_t^i\right)\beta\left(e_t^i\right)z_t\left(e_t^i\right), \qquad \qquad ...(3.4.1)$$

which by definition is equivalent to

$$q\left(z\right) = \rho_0\left(t\right) E\left[z.\beta\right], \qquad \qquad ...(3.4.2)$$

where $E\left[.\right]$ denotes the expectation operator. Hence

$$q\left(z_t\right) = \rho_0\left(t\right)\left\{E\left[z_t\right]E\left[\beta_t\right] + cov\left[z_t, \beta_t\right]\right\}. \qquad \qquad ...(3.4.3)$$

Let $\sigma\left[.\right]$ denote the standard deviation and $r\left[.,.\right]$ denote the coefficient of linear correlation, so that $cov\left[z_t, \beta_t\right] = r\left[z_t, \beta_t\right]\sigma\left[z_t\right]\sigma\left[\beta_t\right]$. Note also that from (3.3.2) $E\left[\beta_t\right] = 1$. Now (3.4.3) becomes

$$q\left(z_t\right) = \rho_0\left(t\right)\left\{E\left[z_t\right] + cov\left[z_t, \beta_t\right]\right\},$$

or

$$q\left(z_t\right) = \rho_0\left(t\right)\left\{E\left[z_t\right] + \sigma\left[\beta_t\right]\sigma\left[z_t\right]r\left[z_t,\beta_t\right]\right\}. \qquad \ldots(3.4.4)$$

Generally, $z_t\left(e_t^i\right)$ may take positive values in some events $e_t^i$ and negative values in other events, so that $E\left[z_t\right]$ may be zero. Prospects in the securities market, however, typically represent cash flows with non-negative contingent cash returns $z_t\left(e_t^i\right)$ (e.g. stocks, bonds etc.) and with positive expected values. Then

$$q\left(z_t\right) = \rho_0\left(t\right)E\left[z_t\right]\left\{1 + \sigma\left[\beta_t\right]\frac{\sigma\left[z_t\right]}{E\left[z_t\right]}r\left[z_t,\beta_t\right]\right\}. \qquad \ldots(3.4.5)$$

Let

$$\alpha\left(z_t\right) = 1 + \sigma\left[\beta_t\right]\frac{\sigma\left[z_t\right]}{E\left[z_t\right]}r\left[z_t,\beta_t\right], \qquad \ldots(3.4.6)$$

and recall the definition (3.1.2.) of $\rho_0\left(t\right)$. Substituting these into (3.4.5) we get the very familiar form of the net present market worth

$$q\left(z_t\right) = \frac{E\left[z_t\right]\alpha\left(z_t\right)}{\left(1 + i_{0t}\right)^t}. \qquad \ldots(3.4.7)$$

The importance of the result here lies in the precise definition in (3.4.6) of the way $\alpha$ depends on the probabilistic characteristic of the cash flow $z$. This may be summarized as follows: there exists a *market variable* $\beta$, such that the worth of any uncertain cash flow $z$ —relative to a riskless cash flow with the same expected value—depends only on the coefficient of variation of $z$, $\sigma\left[z\right]/E\left[z\right]$ and the coefficient of linear correlation of $z$ with $\beta$, but not any other moments (and thus, for example, also not on the skewness of the distribution of $z$). All uncertain cash flows uncorrelated with $\beta$ are evaluated at the riskless rate. *The result does not depend on the rist attitudes of agents, their probability assessments, or the distribution of wealth among investors.*[6]

## 3.5    "Risky" Interest Rates

In the terminology of business finance it is customary to condider the "cost of risky capital" as the interest rate at which the expected value of an uncertain cash flow should be discounted to get its present worth. That is, given a single uncertain cash flow $z_t$ with given worth $q\left(z_t\right)$, we define the risky rate $k\left(z_t\right)$ applicable to $z_t$ by

$$q\left(z_t\right) = \frac{E\left(z_t\right)}{\left[1 + k\left(z_t\right)\right]^t}. \qquad \ldots(3.5.1)$$

Note, that the uncertain cash flow $z_t$ and its worth $q\left(z_t\right)$ are not sufficient to define the applicable rate $k\left(z_t\right)$. Since $k\left(z_t\right)$ is defined by (3.5.1), where $z_t$ appears through $E\left[z_t\right]$, the measure $\pi$ used to evaluate $E\left[z_t\right]$ is implicit in the

---

[6]See also Sharpe [13] and Lintner [9]. Our result here resembles that of Sharpe in the evaluation of "residual" or "unsystematic" variances. The relationship between and both Sharpe's and Lintner's "market return" will become clear in section V.

definition of $k\left(z_t\right)$. Even if there is perfect information on the prospect specification of $z_t$, there need not be agreement among investors as to the appropriate rate $k\left(z_t\right)$, since individuals may differ in their assessments of probability. Unlike the riskless rate $i_{0t}$, which is a market variable, the risky rate $k\left(z_t\right)$ is defined *relative to a probability measure* $\pi$. The same is true, of course, of the coefficient $\alpha\left(z_t\right)$.

Combining (3.5.1) with (3.4.7) we have the identity

$$\frac{\alpha\left(z_t\right)}{\left(1+i_{0t}\right)^t} = \frac{1}{\left[1+k\left(z_t\right)\right]^t}$$

or

$$1+k\left(z_t\right) = \left(1+i_{0t}\right)\left[\alpha\left(z_t\right)\right]^{-1/t},$$

and thus the "risky rate" $k\left(z_t\right)$ is greater, equal, or smaller than the riskless rate $i_{0t}$ according to whether $\alpha\left(z_t\right)$ is less, equal or greater than 1, or, equivalently, according to whether $z_t$ is negatively correlated, uncorrelated, or positively correlated with $\beta_t$. Clearly in the degenerate case when $\beta_t$ is constant (and then necessarily equal to 1) $\alpha\left(z_t\right) = 1$ and $k\left(z_t\right) = i_{0t}$ for all $z_t$, which makes the case irrelevant for our interests. It is easy to see that in all other cases all three relations between $k\left(z_t\right)$ and $i_{0t}$ hold.

# 4   The Economically Relevant Partition of the State-Space

In some cases it is more convenient to use a "state-space-model" for the evaluation of prospects, rather than the concise representation in terms of the probabilistic properties of these prospects. Recall, however, that the events $e_t^i$ in our original state-space have been assumed to represent all information available at time $t$, and that the cash flows of *all* prospects, contingent on those events, are therefore well defined numbers. The implications of this requirement on the dimension of the state-space has been our main motive for the use of probabilities. If a "state-space-model" is to be used, it is interesting to investigate whether the complete listing of $\left\{z\left(e_t^i\right)\right\}$ can be somehow avoided, using a coarser partition and less data. We shall now see that the detailed partition of the state-space is "economically irrelevant", and that a coarser partition can indeed be used effectively. This partition is achieved by lumping together, for each $t$, all states $e_t^i$ that have the same value of $\beta\left(e_t^i\right)$. Formally, let for any $t \in T$

$$\left\{\Omega_t^j, \quad j = 1...m_t\right\}$$

be a partition of $S$, such that

$$\text{if} \quad e_t^i \in \Omega_t^j \quad \text{then} \quad e_t^l \in \Omega_t^j \quad \text{if and only if} \quad \beta\left(e_t^l\right) = \beta\left(e_t^i\right).$$

Note that $\beta\left(\Omega_t^j\right) = \beta\left(e_t^i\right)$ for $e_t^i \in \Omega_t^j$ is well defined. Recall that any prospect $z$ can be written as the sum of single-cash-flow-prospects $z_t$ occurring each at some time $t$ and that, by linearity

$$q\left(z\right) = \sum_{t \in T} q\left(z_t\right).$$

Now recall (3.4.1),

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{i=1}^{K_t} \pi\left(e_t^i\right) \beta\left(e_t^i\right) z\left(e_t^i\right),$$

which can be rewritten as

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{j=1}^{m_t} \beta\left(\Omega_t^j\right) \sum_{e_t^i \in \Omega_t^j} \pi\left(e_t^i\right) z\left(e_t^i\right),$$

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{j=1}^{m_t} \beta\left(\Omega_t^j\right) \pi\left(\Omega_t^j\right) \sum_{e_t^i \in \Omega_t^j} \pi\left(e_t^i \mid \Omega_t^j\right) z\left(e_t^i\right),$$

$$q\left(z_t\right) = \rho_0\left(t\right) \sum_{j=1}^{m_t} \beta\left(\Omega_t^j\right) \pi\left(\Omega_t^j\right) E\left[z_t \mid \Omega_t^j\right]. \qquad ...(4.1)$$

where $\pi\left(e_t^i \mid \Omega_t^j\right)$ is the conditional probability of $e_t^i$, given $\Omega_t^j$, and $E\left[z_t \mid \Omega_t^j\right]$ is the conditional expectaton of $z_t$, given $\Omega_t^j$. Now (4.1) is analogous to (3.4.1), and the *mega-events* $\Omega_t^j$ replace the events $e_t^i$. The significant feature in this coarse partition is that any variation of $z_t$ within a mega-event does not affect the worth, as long as the conditional expecation is unchanged. Within the mega-events, then, the expected value acts as an *effective-certainty-equivalent*. It is easy to see that the suggested partition is the coarsest partition that satisfies this, and it can therefore be termed the *economically relevant* partition of the state-space.

## 5   Market Risk Aversion

### 5.1   Risk Aversion in a Market Sense

The analysis above served to indicate the importance of the variable $\beta$ for the valuation of prospects. Note, however, that $\beta$ is not directly observable, but a composition of the market weights $\nu$ and the probabilities $\pi$. The technical difficulties in inferring $\beta$ on one hand, and its prime role in the worth of capital assets on the other, thus suggest the desirability of an effective economic observable surrogate of $\beta$.

Recall the special case described above where all agents are the same. We have noted in (3.3.4) that in this case $\beta\left(e_t^i\right)$ is proportional to the marginal

utility, and thus a well-defined function of wealth. If, in addition, investors are also assumed risk averse, $\beta$ decreases as $\omega$ increases, with $\beta\left(e_t^i\right)$ highest for the most adverse events and lowest for events of greatest affluence.[7]

This relation between $\beta$ and $\omega$ may hold even if all restrictive assumptions concerning the market are dropped. This may be informally interpreted as though market behaviour is being dominated by risk averse investors, who behave "rationally" with respect to the measure $\pi$. If we wish to restrict the analysis to market variables, we may avoid any statement concerning the behaviour of individuals, and use the relation between $\beta$ and $\omega$ to define *risk aversion in a market sense*: a market, as represented by the worth functional $q$, is said to be *risk averse* (relative to a given measure $\pi$) if, for all $t$, $\beta$ is monotone decreasing with the overall level of social affluence[8] under the various events at that time.

Some words of caution may be in order here. First, note that avoiding any statement about individuals in the definition of risk averse marekts does indeed leave the term non-restrictive in this respect. In fact *all* agents in a risk averse market may be risk seekers, and, conversely, *all* agents may be risk averse and the market may still fail to qualify as risk averse. These extreme situations may, however, be ligitimately considered as far fetched. Second, note that the definition of risk averse marekts does not involve a comparison of social affluence at different time points, but only over different events at the same time. Finally, we may take into account different histories preceding different events by a sufficiently flexible definition of affluence. If the effect of the history preceding a certain time point on the attitudes of (influential) investors towards their marginal dollar income at that time is significant, then the definition of affluence may not be restricted to spot variables such as the G.N.P., but must also include some summary of the relevant history. The operational choice in every specific case will naturally depend in part on empirical considerations.

Some well known phenomena tend to support the position that most observed capital markets tend to satisfy our definition of risk averse marekts. First note that a state of overall affluence is defined as an event under which most investmetns in real assets pay off highly. Therefore, by definition, most investments and corporate securities have higher returns in affluent events than in adverse events. If the market is risk averse, then the returns of most of these assets will be negatively correlated with $\beta$, their certainty equivalents lower then the expected values, and their worth lower than that of riskless returns with the same expected value. The fact that the worth of most risky assets is indeed lower than the worth of riskless assets with the same expected return, or equivalently that the "required rate of return" on risky assets is predominantly higher than the riskless rate thus supports the hypothesis of risk averse markets.

Secondly, consider the opposite phenomenon of prospects with higher returns

---

[7] We shall deliberately avoid the issue of precise definition, or measurement, of social affluence.

[8] The precise operational meaning of the definition of market risk aversion will therefore depend on the precise scale chosen to represent social affluence.

primarily in adverse events. This is basically a phenomenon of "insurance" — more widely recognized on the level of individuals than of society as a whole. Flood control systems, the storage of food reserves, or air-raid shelters all fall in that category. In a risk averse market, investments of this type will be undertaken even if their expected yield is lower than the riskless rate. It is well known that in practice they are indeed frequently undertaken.[9]

## 5.2   Surrogates of $\beta$

Any observable variable may serve as a surrogate of $\beta$ if it can conveniently replace $\beta$ in (3.4.6). Thus, the power of some variable $H$, say, as a surrogate of $\beta$ is determined by the validity of the equation

$$q\left(z_t\right) = \frac{1}{\left(1 + i_{0t}\right)^t} \left\{E\left[z_t\right] + A\sigma\left[z_t\right] r\left[z_t, H\right]\right\}, \qquad ...(5.2.1)$$

where $A$ is a constant. $H$ is a *perfect surrogate* of $\beta_t$ if (5.2.1) holds for all *conceivable* prospects $z_t$.

In a risk averse market, it is natural to search for some observable measure of social affluence as a surrogate of $\beta$, since a monotone relation is presumed. Now if, for some measure $H$ of social affluence, the monotone relation is also linear, namely if

$$\beta = a_0 - a_1 H, \qquad ...(5.2.2)$$

then clearly $H$ is a perfect surrogate. It can be shown without difficulty that (5.2.2) is also a necessary condition for a perfect surrogate. In the restrictive context of (3.3.4), the linearity in (5.2.2) is equivalent to the utility function being quadratic in $H$.

# 6   Concluding Remarks

One of the basic issues of corporate policy under uncertainty may be summarized as follows: given a perfect market where all agents are risk averse, should a corporation, wishing to maximize its market worth, adopt a risk averse policy—reflecting the tastes of its shareholders—or a risk neutral policy—taking into account the fact that shareholders can always diversify their own investments.

---

[9]See, for example, also Arrow [1] and Hirshleifer [8]. The fact that "social insurance" investments are in practice undertaken almost exclusively by government and do not stand to the market test is disturbing. A possible reason is that in states of extreme distress "the rules" change. It is inconceivable that society would tolerate private control of a vital commodity under "national emergency" and calmly pay "supply and demand" prices rather than use the power of law and police, and perhaps pay a compensation in terms of cost. Thus, a police force is an integral part of any "social insurance investment", and since the police force is an exclusive prerogative of government, so are these investments. The fact that they are not traded in the market is in no way an indication of their value in terms of market worth as prospects.

One answer indicated by our analysis is that corporate risk averse behaviour in the context of worth maximization is never "variance avoiding" *per se* . Essential to the notion of worth maximization is not the reduction of overall corporate variance but of best "fit" to the cost of capital $q$. This agrees of course, with the notion of stockholders' diversification of investments: even if shareholders wish to avoid the overall variances in their portfolios, they need not be concerned with the variance of any one particular firm.

The other part of the same answer indicates that shareholders' investment diversification does not imply corporate "expected-value-maximization": the worth of capital assets was shown to be not independent of the variance. In terms of shareholders' preferences this means that there are some types of variances from which shareholders cannot effectively protect themselves by diversification: this is the social risk, as opposed to corporate risk. In broad terms, then, corporate behaviour in perfect capital markets should be averse to social risk, but indifferent to any corporate risks beyond the social risk.

In a risk averse market, we can be more specific in the identification of the two risks. The analysis of section V indicates that corporate policy should be indifferent to variances originating from transfer of assets from one corporation to another, but averse to variances in the utilization of natural resources. Corporations should thus perhaps tend to maximize expected value in problems such as inventory control, maintenance procedures etc., but tend to be more "conservative" in substantial expansion of production capacity, which contributes considerably to overall social wealth.

# References

1. Arrow, K. J. "Discounting and Public Investment Criteria" Western Resources Conference, Seminar in Water Resources Research (July 1965) (mimeo).

2. Arrow, K. J. "The Role of Securities in the Optimal Allocation of Risk-Bearing", *Review of Economic Studies*, **31** (1963–64), 9l–96.

3. Debreu, G. *Theory of Value* (New York, John Wiley and Sons, Inc., 1959).

4. Diamond, P.A. "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty", *American Economic Review*, **57** (1967), 759–778.

5. Green, H.A.J. "Uncertainty and the Expectations Hypothesis", *Review of Economic Studies*, **34** (1967), 387–398.

6. Hicks, J.R. *Value and Capital* (Oxford University Press, 1939).

7. Hirshleifer, J. "Investment Decision Under Uncertainty: Choice Theoretic Approaches", *Quarterly Journal of Economics*, **79** (1965), 509–536.

8. Hirshleifer, J. "Investment Decision Under Uncertainty: Application of the State Preference Approach", *Quarterly Journal of Economics*, **80** (1966), 252–277.

9. Lintner, J. "Security Prices, Risk and Maximal Gains from Diversification", *Journal of Finance*, **20** (1965), 387–616.

10. Lutz, F. "The Structure of Interest Rates", *Quarterly Journal of Economics*, **55**, (1940), 36–63.

11. Myers, S. *Effects of Uncertainty on the Valuation of Securities and the Financial Decision of the Firm* (Ph.D. Thesis, Stanford University, 1967).

12. Radner, R. Competitive Equilibrium Under Uncertainty, *Econometrica* (January 1968).

13. Sharpe, W.F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, **19** (1964), 425–442.