COOPERATIVE GAMES IN EFFECTIVENESS FORM

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1 Introduction

The primary vehicle for the analysis of cooperative games has been the characteristic-function form. This paper seeks to point out certain deficiencies inherent in a characteristic-function type of representation for some cooperative games. A more general form, the effectiveness form, is proposed to correct the inadequacies of the characteristic-function representation. In addition, concepts of stability for effectiveness-form games are suggested; and some examples are studied.

Section 2 establishes notation and introduces some basic definitions and references. In Section 3 the rationale for the effectiveness form is discussed. A formal description of the form is presented in Section 4. Section 5 deals with some possible concepts of stability. Section 6 analyzes a few examples of interest.

2 Preliminaries

Let N be the set of participants in an n-person, cooperative (i.e., enforceable agreements may be made) decision situation. A strategy set $\sum_S$ is associated with each coalition $S \subseteq N$. ($\sum_S$ is usually the set of possibilities available to S through joint randomization over its individual pure strategy sets.) Let P be any partition of N and p be the number of sets composing P. For every p-tuple of strategies $(\sigma_1, ..., \sigma_p)$ with $\sigma_S \in \sum_S$, $S = 1, ..., p$, there is associated an n-tuple of utility payoffs. The above information, for each partition P, constitutes the normal form of a cooperative game.

For each $S \subseteq N$, let E^S denote the Euclidean space of dimension equal to the number of players in S. A point in E^S is a utility payoff to the coalition S, each individual's payoff being denoted by a particular component. The characteristic

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function v maps each coalition $S \subseteq N$ ($S \neq \phi$) into a subset $v(S)$ of E^S. Normally, $v(S)$ is taken to represent a set of utility vectors, each of which the coalition S can "assure" itself in some sense. A game in characteristic-function form (cf form) is a triple (N, v, H) where H represents the set of possible utility outcomes for the players. H is therefore a subset of $v(N)$. Additional assumptions are generally required of (N, v, H). The interested reader is referred to [1] for a more complete discussion of the cf form and for a description of cf-form stability concepts.

As models of cooperative decision situations, the normal and cf forms differ sharply in emphasis. The normal form provides for an explicit statement of strategic possibilities; while the cf form is concerned with the power of a coalition, which is presumably derived from some consideration of the strategies available. In practice, at least two possibilities exist for deriving cf-form games from normal-form games. The usual α derivation places those vectors x in $v(S)$ which the coalition S can assure itself by appropriate strategy choice, no matter what strategies the players in $N \backslash S$ select. The β derivation places those vectors x in $v(S)$ such that the coalition $N \backslash S$, with any of its strategies, cannot prevent S from getting at least x. In general, these derivations may lead to different cf forms. (See [1, p. 20] for an example.)

For any vector $x \in E^N$, let x^S denote the projection of x on E^S. Domination may be defined as follows. For $x, y \in E^N$, x dominates y via the coalition S (xD_Sy) if $x^S \in v(S)$ and $x^S > y^S$ (meaning $x^{\{i\}} > y^{\{i\}}$ all $i \in S$); and x dominates y (xDy) if xD_Sy for some $S \subseteq N$. Domination is at the heart of much of stability theory for cf-form games. The core of (N, v, H) is the set of vectors in H which are undominated by other vectors in H. Von Neumann-Morgenstern solutions [1,4] are also based on the domination relation. Other solution concepts related to those which are to be proposed here are the various bargaining-set [2, 3] and self-policing solutions [6].

3 Motivation

We shall concentrate on two features of cooperative decision situations with which the cf form seems to deal inadequately. The first of these is the possibility that some of the strategies of the normal form have a conditional feature about them which can not be reflected in a characteristic function of the usual variety. The following example (due to John Chamberlin) is illustrative.

Example 1: A community consists of two people The first is a producer who pollutes the environment when engaging in the production process. Player 2 is a composite representing the local legislative body of the community. Possessing the interests of the community, player 2 receives disutility from pollution caused by player l's production. For simplicity, assume that production at level y yields player 1 net utility of y and player 2 net utility of $(-y)$. The technology available to 1 is such that he can produce at any nonnegative level. Player 2 can pass a law outlawing pollution altogether, but the legislative machinery is expensive

		Player 2	
	Production	No Law	Law
	0	(0,0)	$(0, -k)$
	.	.	.
	.	.	.
	.	.	.
	y	$(y, -y)$	$(0, -k)$
Player 1	.	.	.
	.	.	.
	.	.	.
	k	$(k, -k)$	$(0, -k)$
	.	.	.
	.	.	.
	.	.	.

Table I

> to set in motion, and passage of a law would cost player 2 k utility units. On the other hand, the players can sign binding agreements with each other at no cost.

In normal form, the strategies available to 2 are to pass a law prohibiting pollution and to take no action. Player 1 can pollute at any legal level. This means that player l's strategies are conditional statements and depend upon the strategy selected by player 2. The possibility of signing binding agreements is not explicitly stated in the normal form of this cooperative game, but it is understood that any possible strategy combination in the game can be achieved if both players agree to sign a contract.

The normal form of the game can be represented concisely as in Table I (where the entries of the array are ordered utility pairs for players 1 and 2, respectively, at the appropriate strategy combinations). It is understood here that 1's strategy set is conditioned on the passage of no law. If 2 passes the law, 1 has no choice but to produce at level zero. Thus, the payoffs if a law is passed are all $(0, -k)$.

In either α- or β- cf form,

$$\begin{aligned} v(\{1\}) &= \{x : x \leq 0\} \\ v(\{2\}) &= \{y : y \leq -k\} \\ v(\{1,2\}) &= \{(x,y) : x + y \leq 0, y \leq 0\} \\ H &= v(\{1,2\}). \end{aligned}$$

By setting $H = v(\{1,2\})$ we allow for the free disposal of utility by either player.

In passing from the normal to the cf form above of this example, an interesting feature of the problem has been lost; namely, the ability of player 1 to produce at positive levels when no low is passed (and no agreements to the contrary are made). It is precisely this feature of the situation which forces player 2 either to pass the law or to come to an agreement with player 1, yet this asymmetry in the roles of the players is not evident from the cf-form description. Any analysis based on the cf form must therefore ignore this asymmetry as well. The core of the cf-form game is easily seen to be $\{(x, y) : x + y = 0,\ 0 \leq x \leq k\}$. Though all points in this set are undominated, the point $(0, 0)$, for example, which can only result when no law is passed, is subject to attack by player 1. In particular, 1 possesses the ability and the desire to produce at a positive level given that no law is passed.

Example 1 seems to suggest that the power of a coalition may not, in general, be expressible in the unconditional sense of the characteristic function, no matter what derivation is used. Rather, it may be preferable in some cases to define the strength of a coalition conditionally on particular strategy combinations.

The second feature of cooperative situations for which the cf form may be inadequate is its restricted view of threat possibilities. This is illustrated below.

> **Example 2:** Three flower-loving people live in separate homes arranged in a triangular pattern. Each person has a garden on his property and derives pleasure from looking out at beautiful flowers from his living-room window. From each living-room window, the view is concentrated mostly on the garden of the neighbor to the left, only somewhat on the person's own garden, and not at all on the garden of the neighbor to the right. Each person owns \$10 worth of fertilizer which is useful only for beautifying gardens. Because of the views involved and the usefulness of fertilizer, each person's utility function may be expressed as being linear in the value of the fertilizer used on each garden. For every \$10 worth of fertilizer used on his own garden, a person receives one utility unit; for every \$10 worth used on his neighbor's to the left, he receives 5 utility units; and for beautification of the garden of his neighbor to the right, he receives no utility at all. Of course each person has the absolute authority to determine what is done to his own garden.

Example 2 is actually an example of a market with production wherein the production process (i.e., fertilization) creates a sizeable external economy for one of the other participants (i.e., the view of the neighbor to the left is improved). Consideration of the cf form of this game leads to some interesting insights concerning the role of threats in the domination relation and the core. Let an arbitrary player be denoted player 1, his neighbor to the left player 2, and his neighbor to the right player 3. Then

$$\begin{aligned} v(\{i\}) &= \{x : x \leq 1\} \text{ for } i = 1, 2, 3; \\ v(\{1, 2\}) &= v(\{2, 3\}) = v(\{3, 1\}) = \{(x, y) : (x, y) \leq (10, 2)\};\ \text{and} \\ v(\{1, 2, 3\}) &= \{(x, y, z) : x \leq 5\beta + \alpha, y \leq 5\gamma + \beta, z \leq 5\alpha + \gamma; \\ &\qquad \alpha + \beta + \gamma = 3; \alpha, \beta, \gamma \geq 0\} \end{aligned}$$

That is, an individual acting alone can guarantee himself no more than a utility of 1; since he may only use his fertilizer on his own garden. Similarly, a

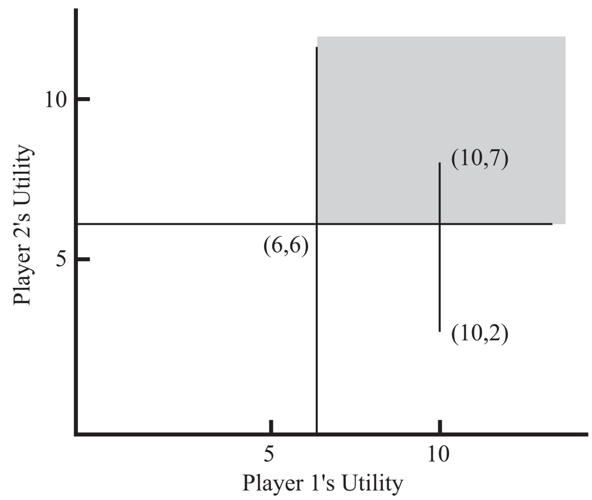

Figure 1: Example 2 in the Utility Space of $\{1, 2\}$

coalition of two players can do no more than to use their combined fertilizer on the garden of the player to the left in the coalition. The cf value for $\{1, 2, 3\}$ reflects its ability to use the combined amounts of fertilizer on the gardens in any manner it pleases.

If each individual uses his own fertilizer on his own garden, the utility outcome is $(6, 6, 6)$, which is clearly in the core of the cf game. Notice, however, that if players 1 and 2 agree to use their combined supply of fertilizer on the garden of player 2 and to accept no fertilizer from player 3, player 3's only rational action is to fertilize his own garden (since no other action yields him as much utility). This outcome yields the utility payoff $(10, 7, 1)$, which players 1 and 2 both prefer. The point $(10, 7)$ does not dominate the point $(6, 6)$ via the coalition $\{1, 2\}$ except through a certain action by player 3, even though this action is his only rational decision. Thus, the use of the domination relation implicitly assumes that players expect other players actually to carry out threats which are injurious to their own welfare (in this case player 3 discarding his fertilizer). In Fig. 1, this particular situation is depicted in the utility space of the coalition $\{1, 2\}$. Here the utility outcome set may be restricted by $\{1, 2\}$ to the line segment $[(10, 7), (10, 2)]$. The domination relation recognizes only the point $(10, 2)$, but the assumption of rationality for player 3 leads to the point $(10, 7)$. The shaded area represents the region preferred by $\{1, 2\}$ to the utility outcome $(6, 6)$.

As is evident in Example 2, the use of cf-form analysis (in particular, analysis based upon the domination relation) implies an acceptance of certain assumptions about the role of threats in the games under consideration. For some games such assumptions may be unimportant or even desirable. From the analyst's point of view, it would be preferable if he could choose for himself, based on the game under study, whether or not and to what extent threats are to be considered.

4 Effectiveness Form

The effectiveness form (e form) provides a framework more general than the cf form for analysis of cooperative decision situations. This framework is general enough to model cooperative games in normal form yet not so general that it precludes a meaningful theory of stability.

A *game G in effectiveness form* consists of:

(a) a set N of players;

(b) a set Q of outcomes;

(c) an ordinal vector-valued utility function $u : Q \to E^N$; and

(d) for each coalition $S \subseteq N$, an *effectiveness function* which maps every point $q \in Q$ into a collection of subsets of Q.

The effectiveness function for S is intended to identify, for any proposed outcome q, the set of alternative outcome subsets which S can enforce against q. This means that when faced with q, the coalition S is able, in some sense, to restrict the negotiation process (at least temporarily) to any one of the specified subsets of the outcome set. S cannot, however, in general, determine which particular outcome in such a specified set will result without the concurrence of members of $N \backslash S$.

In Example 1, player 1, faced with a law, has no alternative but to produce at level zero and hence is effective for nothing more. Faced with a proposal involving no law, he can produce at any nonnegative level and thus is effective for any set of outcomes in Q with which no law is associated. This is a very temporary type of effectiveness, however, since player 2 is effective to pass the law against any such single outcome. In Example 2 the effectiveness functions do not depend on any specified proposals. In particular, the various coalitions are effective to allocate their own endowments of fertilizer on gardens of members in any manner they choose. Such effectiveness is independent of any point q proposed and therefore holds for all points q in Q.

The expression XE_Sq will mean that X is a set of outcomes to which S can restrict the game (in the sense described above) when faced with the proposal q. We adopt the convention that $XE_\phi q$ is false for all $X \subseteq Q, q \in Q$. In e-form games derived from cooperative decision situations, the intuitive meaning of effectiveness leads to certain natural conditions on the effectiveness relations:

(1) If $X_1 E_S q$ and $X_2 E_S q$ then $(X_1 \cup X_2)\, E_S q$.

(1′) If XE_Sq and $X \subseteq Y \subseteq H$, then YE_Sq.

(2) If $S_1 \subseteq S_2 \subseteq N$, then $XE_{S_1}q$ implies $XE_{S_2}q$.

(3) If XE_Sq and $q \notin X$, then there is some $x \in X$ such that $\{q\}E_M x$ is false for all $M \subseteq N \backslash S$.

Condition (1) and the stronger version (1′) reinforce the idea that the coalition S can restrict the negotiations to X but cannot, in general, determine specific outcomes in X. One could also extend Condition (1) to include infinite unions. Condition (2) is similar to the common superadditivity assumption in cf games. Its effect is that addition of players to a coalition does not weaken the coalition. Condition (3) states that if a coalition S can enforce a move away from q, then for at least one outcome which might result, no subset of $N \backslash S$ can enforce a return to q.

The e form has the potential flexibility to deal with the previously noted deficiencies of the cf form. The power of a coalition is not independent, in general, of the particular outcome under consideration in this form. Thus, conditional effectiveness can be dealt with. Furthermore, since coalitions may only be effective for sets of outcomes instead of particular outcomes, some flexibility in handling threats is also gained. At least the analyst can identify those threats he deems realistic within the framework of the e form rather than having them dictated as in the cf form for Example 2. (Of course, no theory based on ordinal utility can deal directly with threats in a satisfactory manner, since "this will hurt you more than me" is not an ordinally recognizable statement. Hence the analyst should be left with the decision as to which threats are in fact believable.)

A well-described cooperative decision problem does not necessarily have a unique e-form description. Rather, the characteristics of the situation and the purposes of analysis should guide the analyst in selecting the particular e form to be used. For a game presented originally either in normal or cf form, however, certain e-form derivations are generally available.

For normal-form games there seem to be two e-form derivations of interest. Let $\mathcal{P}$ be the set of all partitions P of the players of N, and let $\sum_S$ be the strategy set for the coalition S. The set Q of possible outcomes is

$$\left(\cup_{P \in \mathcal{P}} \times_{S \in P} \sum\nolimits_S \right).$$

If q is an outcome, then q^S denotes that part of the strategy combination played by S. (If, for example, q is jointly randomized over the players of N, q^S becomes the part of the randomization by the players of S alone.) Suppose that a point $q \in Q$ is proposed. The coalition S is able to alter its strategy choice to any set $T_S \subset \sum_S$. There are two options available to express this:

$$\left(T_S \times \sum\nolimits_{N \backslash S} \right) E_S q; \tag{1}$$

and

$$\left(T_S \times q^{N \backslash S} \right) E_S q. \tag{2}$$

The first option says that the coalition S may change its strategy but so may the complementary coalition. In this case the effectiveness relations are independent of the proposed q. The second option says that the coalition S may change its strategy, but the complementary coalition must retain its proposed strategy. The latter is to be interpreted as merely a temporary state of affairs; since for any specific point in $\left(T_S \times q^{N\setminus S}\right)$ the complementary coalition, or any other coalition, is effective to change its own strategy selection. With the appropriate option selected the effectiveness form is completely determined, since the player set and the utility functions are taken directly from the normal form. Conditions $(1')$, (2), and (3) are all satisfied under both options.

In Example 1, suppose q is an outcome at which no law has been passed. Then in deriving the e form from the normal form under option 2, player 1 is effective to produce at any level or to select any set of production levels. Option 2 is therefore useful in correcting this deficiency of the cf form for this example, since player 1's conditional effectiveness is explicitly stated. The viewpoint of option 1 is not without intuitive appeal, however, since the possible retaliatory actions of the complementary coalition are carried along in the effective sets and are therefore available for use. In Example 2 if, at q, each player fertilizes his own garden; then, if two players form a coalition, the threat possibilities of the third are included in the effective sets under option 1 but not under option 2; since in the latter case he continues, at least temporarily, to play his part of q.

For games in cf form, the e-form derivation is made under the assumption that the cf form is a reasonable description of the decision problem at hand. Thus, if $x^S \in v(S)$, S can take some joint action which yields itself at least x^S. For this reason, it is convenient to exclude games in which $H \neq v(N)$. If S takes an action which yields itself at least x^S and $N\setminus S$ takes an action which yields itself at least $x^{N\setminus S}$ but $\left(x^S, x^{N\setminus S}\right) \notin H$, then the cf form does not describe what the utility outcome of such actions is to be. On the other hand, if $H = v(N)$, the players can assume that the resulting utility outcome is $\left(x^S, x^{N\setminus S}\right) \in v(N) = H$. With the provision that $H = v(N)$, the e-form derivation becomes similar in spirit to option 1 for normal-form games.

Let x be any vector in $v(S)$. Let $K_{S,x} = \{y \in H : y^S \geq x,\ y^{N\setminus S} \in v(N\setminus S)\}$. From our intuitive discussion of effectiveness, we should require $K_{S,x} E_S h$ all $h \in H$, all $x \in v(S)$. That is, if a vector x is in $v(S)$, then S can assure that an outcome which yields itself at least x occurs; and the members of $N\setminus S$ can do nothing to yield themselves a payoff outside of $v(N\setminus S)$. In addition, one can either allow unions (finite or infinite) of sets of the form $K_{S,x}$ where $x \in v(S)$ or supersets which are contained in H. It is clear that Conditions (2) and (3) are satisfied for this derivation as long as two standard assumptions about cf games are made; namely,

$$(i)\ v(S \cup T) \supseteq v(S) \times v(T) \text{ if } S \cap T = \emptyset, S \cup T \subseteq N; \text{ and}$$

$$(ii)\ v(N) = \{x \in E^N : \exists y \in H \text{ such that } y \geq x\}.$$

It should be evident by now that the effectiveness form is designed for the evaluation of proposed outcomes in terms of what alternatives might be offered as counterproposals. Thus, we are implicitly assuming some preplay negotiation process from which the outcome or set of outcomes is to be selected. The rules governing this negotiation process are, of course, critical in determining the outcome or outcomes selected. Each of the solution concepts to be described implicitly assumes particular standards, but the overall structure of the negotiation process is assumed to be that of a proposed outcome or outcome set being subjected to attack (or objection) by coalitions which propose other outcomes. The attacks may themselves be subjected to some further consideration (counterattack or counterobjection). Both the attack and the counterattack must be made according to whatever standards are adopted.

There is no reason to suppose that this or any other view of a preplay negotiation process possesses any universal validity. Nevertheless, models of this sort have interested theorists in the past (e.g., von Neumann-Morgenstern solutions and bargaining sets seem to have similar underlying motivations). The e form simply provides a vehicle for examinations of this sort which eliminates the deficiencies (discussed in the two examples above) inherent in a cf-form approach.

5 Stability Concepts

Several notions of stability have appeared in the literature of cooperative games. The kind of stability to be considered here is similar to both the bargaining-set concept (from which the terminology is borrowed) and the self-policing type of stability. The notions are intended to be merely suggestive of the sort of ideas which can be incorporated within the e-form framework.

Suppose an outcome q arises in a negotiation process. The coalition S is effective for some set X against q. Furthermore, let X' be those points in X which might "reasonably" arise (other points in X involve "unreasonable" threats by members of $N\backslash S$). Then S *objects* to q with *objection set* X if every player in S prefers every point in X' to q. X' is called the *prime objection set* for this objection. One possibility for X' might be the subset of X consisting of those outcomes with utility images which are Pareto optimal for $N\backslash S$ in the utility image of X and against no point of which any individual acting alone has both the power and incentive to move. The *core of an e-game* is defined to be the set of outcomes against which there exist no objections. This rather strong type of stability is obviously motivated by the analogous cf-form concept. In fact, if a cf game satisfies superadditivity [(i) above] and has $H = v(N)$, its core is clearly identical to that of the e-form game derived earlier [with either Condition (1) or (1′)].

For games with empty cores, we are interested in a weaker type of stability. If a coalition objects to an outcome, its prime objection set may be attacked in turn by some newly formed coalition. We consider one formalization of this idea. A coalition T *counterobjects* to objection (X, X') by S with *counterobjection set* Y and *prime counterobjection set* Y' if (Y, Y') forms an objection by T to some point in X'. Furthermore we require $Y \not\subseteq X'$; i.e., Y is not simply some selection from the points in X'. A counterobjection (Y, Y') by T is *punishing* if for every point y in Y', there is a player in S (the objecting coalition) who prefers q (the initial outcome) to y. Intuitively, existence of punishing counterobjections would tend to deter objections. A counterobjection (Y, Y') by T is *retentive* if each player in T is at least as well off at every point in Y' as he is at q. Retentive counterobjections are similar in spirit to the counterobjections in the various bargaining-set definitions (See [2].)

Stable sets of various types can now be defined. A set R in Q might be called stable if for every objection there were a punishing (or retentive) counterobjection. One could also require that the prime sets of such counterobjections be subsets of R. In fact, a hierarchy of such solution concepts is possible. (See [5].)

One difficulty with the concepts above is that objection sets might exist whose prime subsets were empty due to lack of Pareto boundaries. This can he circumvented by only permitting coalitions to be effective for sets whose utility images are closed and bounded above. Such a restriction would not seem to be serious in applications. Another difficulty is that stable sets may contain points which actually do not "contribute" to the stability of the set. Hence, we tend to be more interested in minimal stable sets; i.e., stable sets no subsets of which are also stable.

In the next section, we seek various minimal stable sets, assuming that coalitions are effective only for sets with utility images which are closed and bounded above.

6 Examples

Example 1. Considering the e-form as derived under option 2 from the normal form, the core is empty. If player 2 objects to a level of production $> k$, $\{1\}$ has no power to counterobject. $\{1, 2\}$ has the power and incentive to counterobject, however, with compact sets of production agreements for any levels p, $0 < p < k$. Such counterobjections are neither punishing nor retentive, however. If $\{1\}$ has an objection to any outcome, the prime objection set must be a production level $\leq k$, since no other points are individually rational for player 2. Such objections by $\{1\}$ cannot be countered, except by himself — in which case such counterobjections would be retentive but not punishing. The set consisting of the two outcomes which yield utility pairs $(0, -k)$ and $(k, -k)$ is a minimal stable set with the property that every objection is countered by a retentive counterojection from the stable set. If $\{1\}$ objects to $(k, -k)$, $\{2\}$ passes a law. 1f $\{1, 2\}$ objects to $(0, -k)$, then $\{2\}$ counterobjects with production at level k. Although the first counterobjection is both punishing and retentive, the second is only retentive.

For the e form derived under option 1, production at level k with no law is an outcome in the core. Should player 1 attempt an objection, the prime set will consist of the point at which a law is passed, making player 1 worse off. Clearly, this is the only point in the core.

Under either option, e-form stability concepts lead to quite different results than the cf-form core for this game. The reader may choose for himself.

Example 2. For the e form derived under option 1, the point at which all players fertilize their own gardens ceases to be in the core. The coalition $\{1, 2\}$ can successfully object by allocating its endowment of fertilizer to the garden of player 2, since player 3's only rational action is to fertilize his own garden. In fact, it can be easily shown that no point is in the e-form core under this derivation.

Consider the allocation of $20 worth of fertilizer to 2's lawn and $10 worth to 3's lawn. The only objections to this outcome are by the coalition $\{2, 3\}$. But any such objection can be countered by $\{1, 3\}$ allocating its endowment to l's lawn. This counterobjection is punishing (for player 2) but not retentive. The three-point set R whose utility image is $\{(10, 7, 2), (2, 10, 7), (7, 2, 10)\}$ is therefore a minimal stable set for which the prime counterobjection sets are contained in R and for which the counter-objections are punishing.

The e-form derivation under option 2 cannot be considered until the normal form is more completely defined. Suppose player 1 allocates his fertilizer to 2's garden at outcome q. If $\{2, 3\}$ considers an objection, the restriction of q to the strategy set of $\{1\}$ must be specified, since 1 can no longer fertilize 2's garden. If the restriction is to fertilize his own garden, then the same analysis applies under either e-form option.

The remaining examples are cf games. The derivation explained earlier will be assumed.

Example 3:

$$\begin{aligned}
N &= \{1, 2, 3\}, \\
v(\{i\}) &= \{x : x \leq 0\}, \\
v(\{1, 2\}) &= v(\{1, 3\}) = v(\{2, 3\}) = \{(x, y) : x + y \leq 1\}, \\
v(\{1, 2, 3\}) &= \{(x, y, z) : x + y + z \leq 1\}, \\
H &= v(\{1, 2, 3\}).
\end{aligned}$$

This familiar coreless game is well discussed in [7]. The only nondiscriminatory von Neumann-Morgenstern solution is

$$V = \{(1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2)\}.$$

V is easily seen to be a minimal stable set with punishing, retentive counter-objections whose prime sets are contained in V. If the retentive requirement is dropped, other three-point sets which are not solutions are found to be minimal stable sets; e.g., $\{(0.6, 0.4, 0), (0, 0.6, 0.4), (0.4, 0, 0.6)\}$.

Example 4:

$$\begin{aligned}
N &= \{1,2,3,4\} \\
v(\{i\}) &= \{x : x \le 0\},\ i = 1,2,3, \\
v(\{4\}) &= \{x : x \le 1\}, \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = \{(x,y) : x+y \le 1\}, \\
v(\{i,4\}) &= \{(x,y) : x \le 0, y \le 1\},\ i = 1,2,3, \\
v(\{1,2,3\}) &= \{(x,y,z) : x+y+z \le 1\}, \\
v(\{1,2,4\}) &= v(\{1,3,4\}) = v(\{2,3,4\}) \\
&= \{(x,y,z) : x+y \le 1, z \le 1\}, \\
v(\{1,2,3,4\}) &= \{(x,y,z,w) : x+y+z \le 1, w \le 1\}, \\
H &= v(\{1,2,3,4\}).
\end{aligned}$$

This game is the same as Example 3 except that player 4, who does not interact with the other three players, is added. This lack of interaction should be reflected by the notions of objection and counterobjection. Consider the outcome $(0,0,0,0)$. Player 4 should certainly be able to object by simply taking one. The prime set for this objection is

$$\{(x,y,z,1) : x+y+z = 1;\ x,y,z \ge 0\}.$$

Intuitively, we should expect no counterobjection to such an objection. In fact, none exist, but only because we have required that the counterobjection set be not wholly contained in the prime objection set. Otherwise $\{1,2\}$ could certainly counterobject to $(0,1/2,1/2,1)$, for example. With player 4 receiving 1 at every point, the minimal stable sets of each type are similar to those of Example 3.

The various minimal stable sets presented here are generally difficult to compute. At present, there seem to be no computationally attractive features of these concepts; and work on the examples above as well as on other games seems to proceed by enumerating all possibilities. It is hoped that further research will result in computationally tractable and intuitively reasonable modifications of the concepts or in the discovery of some heretofore unnoticed mathematical niceties.

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