ON THE ROLE OF A MONEY COMMODITY IN A TRADING PROCESS

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Abstract

An exchange economy is considered, where commodities are exchanged in subsets of traders. No trader gets worse off during the process. As shown by counterexample, the process may converge to a non-Pareto-optimum when a money commodity assumption is dropped.

1 The Exchange Economy ...

Consider an exchange economy of the following type: There are n traders and m commodities, with $n > (m+1)$. The initial endowment of trader j is $w_j \in R^m_+$. It is assumed that $\sum_{j=1}^n w_j > 0$. An allocation $x = (x_1, x_2 \ldots x_n)$ is an element of $\{x|x \in R^{mn}_+, \sum_{j=1}^n x_j = \sum_{j=1}^n w_j\}$. For each trader, there is a relation D_j, defined on R^m_+ and read 'as desirable as'. From D_j, one can define a relation P_j on R^m_+, read 'strictly preferred to', as follows: For all x_j, $y_j \in R^m_+$, $x_j P_j y_j$ if and only if not $y_j D_j x_j$. It is assumed:

A.1. D_j is complete, reflexive, and transitive.
A.2. Convexity: For all x_j, $y_j \in R^m_+$, if $x_j P_j y_j$ then $(\lambda x_j + (1-\lambda) y_j) P_j y_j$ for all $\lambda \in (0, 1)$.
A.3. Free disposal: For all x_j, $y_j \in R^m_+$, if $x_j \geq y_j$, then $x_j D_j y_j$.
A.4. Continuity: For all $y_j \in R^m_+$, the sets $\{x_j|x_j D_j y_j\}$ and $\{x_j|y_j D_j x_j\}$ are closed.

Let G be some group of traders. An allocation x is Pareto-optimal for G if, for any other allocation y, $y_j P_j x_j$ for any $j \in G$ implies $x_i P_i y_i$ for some $i \in G$. An allocation is k-way Pareto-optimal if it is Pareto-optimal for all groups of size k or smaller. An allocation is thus over-all Pareto-optimal if it is n-way Pareto-optimal.

In what follows, we will be interested in trading processes with the following properties: (a) Each trade involves only a subset of the total set of traders, (b)

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no trader gets worse off at any stage of the process, and (c) through a sequence of trades involving subsets of traders, the economy should move to an over-all Pareto-optimal allocation. Such processes have been studied by, e. g., Feldman (1973), Graham et al. (1976), and Madden (1975). Similar investigations with applications to primal decomposition in mathematical programming have been undertaken by Jennergren (1979) and Polterovich (1970).

2 ... with and without a Money Commodity

In this note, interest focuses on the importance of a money commodity in the exchange economy. For the moment, the following assumption is also imposed:

A.5. (a) For all x_j, $y_j \in R_+^m$, if $x_j \geq y_j$ and $x_{jm} > y_{jm}$, then $x_j P_j y_j$. (b) At any point in the trading process, $x_{jm} > 0$ for all traders j. Also, $w_{jm} > 0$.

The mth commodity may be referred to as a money commodity. Apparently, every trader always likes more of commodity m and always has a positive quantity of it.

Under Assumptions A.1 - A.5, it holds

Proposition 1. *m-way Pareto optimality implies over-all Pareto optimality.*

One can show that k-way Pareto optimality does not in general imply over-all Pareto optimality, if $k < m$. Consider now a trading process based on trading groups of size m. There are $Q = \binom{n}{m}$ possible different trading groups of size m that can be formed. Denote these I^1, I^2 ... I^Q. Let trading group I^1 meet. If the current allocation is not Pareto-optimal for I^1, then the members of I^1 effect a trade leading to an allocation which is Pareto-optimal for I^1 and which provides no member of I^1 with a commodity vector less preferred than the one he had before. Then trading group I^2 meets, etc. A series of such meetings, one for each trading group I^1, I^2 ... I^Q, is called a cycle. The trading process goes on for cycle after cycle, until a cycle results in no reallocation. At that point, the process stops.

Proposition 2. *The trading process has one or more limit points. All limit points are over-all Pareto-optimal allocations, and all traders are indifferent among them.*

Propositions 1 and 2 can be proved using arguments from Graham et al. (1976). However, A.5 was not imposed in that paper. Rather, a stronger version of A.3 was used: For all x_j, $y_j \in R_+^m$, if $x_j \geq y_j$ and $x_j \neq y_j$, then $x_j P_j y_j$ (non-satiation).

Suppose now that the money commodity no longer exists; i. e., A.5 is deleted. One can show (see Madden (1975)):

Proposition 3. *$(m+1)$-way Pareto optimality implies over-all Pareto optimality.*

Consider therefore a trading process with trading groups of size $(m+1)$. Let $R = \binom{n}{m+1}$ be the number of different trading groups of that size. Denote those groups J^1, J^2 ... J^R. In this case, a cycle consists of a series of one meeting each for J^1, J^2 ... J^R. Whenever a trading group meets, a reallocation within the group is carried out as before, i. e., the members effect a trade leading to an allocation which is Pareto-optimal for the group and makes no member worse off.

Unfortunately, and this is the main point of this note, the convergence proof breaks down for this second trading process. In particular, the point-to-set map which takes the allocation existing at the outset of one cycle into allocations at the end of the same cycle is no longer closed. This suggests that the process may converge to an allocation which is not over-all Pareto-optimal. The counterexample to follow shows that this is, indeed, so.

3 A Counterexample without Money Commodity

Let $n = 7$ and $m = 4$. Each trader's initial endowment and induced utility function are as follows:

	Initial endowment	Utility function
$j = 1$	$(1,\ a,\ 0,\ 0)$	$x_{11}x_{12} + x_{13}$
$j = 2,\ 3\ ...\ 6$	$(0,\ 1 - 0.2a,\ 1 + 0.2a,\ L)$	$10x_{j2} - x_{j2}^2 + 10x_{j3} - x_{j3}^2 + x_{j4}$
$j = 7$	$(1,\ 0,\ 0,\ 0)$	x_{71}

L denotes a 'large', but unspecified, amount of commodity 4. As regards a, it is assumed: $0 < a \leq 1$. These utility functions agree with Assumptions A.1 - A.4 (if the utility functions for traders $j = 2,\ 3\ ...\ 6$ are appropriately redefined for $x_{j2} > 5$ and $x_{j3} > 5$, but that is without importance in this example). In this case, $R = 21$. Let the trading groups J^1, J^2 ... J^{21} be specified as in Table 1.

It is now assumed that the resulting trading process is such that trader 1 is prepared to exchange commodity 2 for commodity 3 at the rate one-to-one (meaning that his utility level remains constant through the process). Under this assumption, trading groups J^1, J^5, J^9, J^{13}, and J^{17} involve such one-to-one exchanges between trader 1 and traders 2 - 6.

Each of the trading groups J^2, J^3, J^6, J^7, J^{10}, J^{11}, J^{14}, J^{15}, J^{18}, and J^{19} results in no reallocation at all (since the allocation is already Pareto-optimal for the particular group).

Trading groups J^4, J^8, J^{12}, J^{16}, J^{20}, and J^{21} result in reallocations among traders 2 - 6. Formally, trader 7 is also a member of the first five of these groups but obviously cannot participate in the trading. Pareto optimality within each group obviously requires that all included traders 2, 3 ... 6 end up with identical amounts of commodities 2 and 3. To assure that no trader ends up with a commodity vector less preferred than the one he had before the trade, there has to be a redistribution of commodity 4 as well. In effect, commodity 4 serves

Table 1
Trading groups of counterexample†

	Traders						
Group	1	2	3	4	5	6	7
J^1	+		+	+	+	+	
J^2	+		+	+	+		+
J^3	+		+	+		+	+
J^4		+	+	+	+		+
J^5	+	+		+	+	+	
J^6	+			+	+	+	+
J^7	+	+		+	+		+
J^8			+	+	+	+	+
J^9	+	+	+		+	+	
J^{10}	+	+			+	+	+
J^{11}	+		+		+	+	+
J^{12}		+		+	+	+	+
J^{13}	+	+	+	+		+	
J^{14}	+	+	+			+	+
J^{15}	+	+		+		+	+
J^{16}		+	+		+	+	+
J^{17}	+	+	+	+	+		
J^{18}	+	+	+	+			+
J^{19}	+	+	+		+		+
J^{20}		+	+	+		+	+
J^{21}		+	+	+	+	+	

†Inclusion of a trader in a trading group is denoted by +.

as a money commodity for traders 2 - 6. Precisely how much of commodity 4 is transferred among members of each trading group need not be specified here. The important point is that whenever traders 2 - 6 or a subset of these traders meet, then they end up with identical holdings of commodities 2 and 3 (*i.e.,* each one of them has the same amount of commodity 2, and similarly for commodity 3), and no trader gets worse off. It may be noted that J^{21} involves all five of traders 2 - 6. Hence, all five have identical holdings of commodities 2 and 3 after each cycle.

The allocation resulting from the first cycle will then be

$j = 1$	$(1,\ a/(5 \cdot 4^4),\ a(1 - 1/(5 \cdot 4^4)),\ 0)$
$j = 2, 3 \ldots 6$	$(0,\ 1 - a/(5^2 \cdot 4^4),\ 1 + a/(5^2 \cdot 4^4),\ L)$
$j = 7$	$(1,\ 0,\ 0,\ 0)$

Hence, the limiting allocation will be

$j = 1$	$(1,\ 0,\ a,\ 0)$
$j = 2, 3 \ldots 6$	$(0,\ 1,\ 1,\ L)$
$j = 7$	$(1,\ 0,\ 0,\ 0)$

which is not Pareto-optimal (trader 1 could without loss transfer commodity 1 to trader 7).

As a final remark, investigations like this one may be of some interest, in that they clarify the role of money in facilitating exchange (cf. also Ostroy (1973) and Ostroy and Starr (1974) for two other studies somewhat similar in spirit).

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