EQUILIBRIUM CONTRACTS FOR SYNDICATES WITH DIFFERENTIAL INFORMATION

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Abstract

This paper proposes a concept of the core for games with differential information, using Aumann's notion of common knowledge. The concept is applied to solve the syndicate problem, for cases in which members have different private information on the uncertain prospects of each syndicate action and the contract (the risk sharing rule and the decision rule) is to be determined before they exchange their information.

1 Introduction

A group of individuals forms a syndicate to make a common decision under uncertainty that will result in a payoff to be shared jointly among them. The formation of a syndicate is typically motivated by the existence of complementary economic resources such as technology, wealth endowment, and information, as well as by the prospect of mutual risk sharing. In order to enhance cooperation among the individuals, they must reach agreement on the terms of a contract specifying the course of action to be taken jointly and a rule for sharing the payoff accruing to the syndicate. This paper provides a conceptual framework to determine the optimal contract.

Wilson's work [20] has been the source of various studies on risk sharing and group decision making. By using Pareto optimality as the normative criterion, he derived the necessary and sufficient conditions for linearity of the sharing rule, and for existence of a group utility function and an aggregation of the members' probability assessments. Rosing [16] and Wallace [19] required a stronger qualification for aggregation of individual preferences in the form of a group

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utility function and group probability assessments: the Pareto optimal utility frontiers corresponding to all possible decisions must not cross. They then independently found that the group choice behavior was consistent with the Savage axioms only in the restrictive situation in which either (i) the individual utility functions have constant *risk tolerances* (i.e., exponential utilities) or (ii) there is agreement on the probability judgment and the individual utility functions have constant and identical *risk cautiousness* (i.e., exponential, logarithmic or power utilities).[1] In addition, Rosing showed that if Nash's bargaining solution is used to determine the particular point to be chosen from each Pareto frontier, then the individual preference orderings of all decisions are identical. This paper is motivated by the more operational interest of solving the problem, independent of whether a nice aggregation theorem is available or not. Hence, we will employ a normative criterion for a solution which is stronger than Pareto optimality.

An important subject to be incorporated into our model is information. It is widely observed that different people assign different subjective probabilities to the same event. One can simply attribute this phenomenon to differences in beliefs or opinions. All the work cited above is based on this assumption, as is, for example, the notion of *bets* among several agents. The view is, in fact, very prevalent in traditional economic theory, not to mention the Arrow-Debreu general equilibrium theory for economies under uncertainty (Arrow [1], Debreu [6]) and Lintner's model of capital asset prices [12].[2]

In contrast, one can take the opposite view that discrepancies among individual probability judgments should be traced exclusively to differences in the information they have acquired about the world as a result of their divergent historical experiences. The probability assessment of each agent, then, could be interpreted as *conditional* probability assessment which is derived from one and the same *prior* probability measure on the underlying state space, conditioned by the information available only to himself. Such a view has been eloquently set forth by Harsanyi [9] in his work on noncooperative games with incomplete information.

These two views have opposite implications when communication is allowed among agents. If different probability judgments are solely due to differences in personal beliefs, then the probability numbers will not be revised by mutual discussions. On the other hand, if different probability judgments are due to differences in information, and if the individuals honestly reveal their information to each other, then, by correct calculations, they will eventually reach agreement on the probability judgment. In an intriguing paper [4] Aumann proved the following statement: If two people have the same priors, and their posteriors for a given event are *common knowledge*, then these posteriors must be equal even though they may base their posteriors on quite different information.[3] This

[1]Wilson [20] calls this the case of a linear, determinate sharing rule.

[2]Radner [15] introduces differential information into his equilibrium model of an economy under uncertainty, but he maintains the assumption that traders come to the market with the same prior information.

[3]The key concept in this statement is *common knowledge*. The definition is given in Section

implies that the process of (honestly) exchanging information on the posterior probabilities of a given event will continue until these values are equal. Aumann suggests this result as possible evidence against the Harsanyi doctrine. It is, in fact, a bitter pill for proponents of the Harsanyi doctrine to swallow, since the exchange of mere probability numbers, not of the full information, is shown to be sufficient to deny the possibility that people having the same priors settle with divergent probability judgments, or that people *agree to disagree*. In this paper we choose not to discuss this philosophical issue but, instead, take an eclectic standpoint; namely, we will allow situations in which different agents have different priors as well as different information.

And yet the assumption of the presence of differential information necessitates introducing a new set of solution concepts that replaces such fundamental cooperative concepts as efficiency (Pareto optimality) and the core. An individual who has superior private information may seek to strengthen his bargaining power by hiding his information, and those who have poorer information may react by insuring themselves against the unknown information. This salient feature of differential information would not be reflected adequately in the solution, if one should simply build the notions of efficiency and the core by taking each agent's preferences as defined by his current information.

In this paper we propose a new concept of the core to cope with this informational problem. The basic assumption of our core concept is that the syndicate members are unwilling to reveal their private information in the negotiation process. This assumption is embodied in our definition of blocking, which involves Aumann's notion of common knowledge. Loosely speaking, we assume that an agent declares his intention to join a blocking coalition only when it is common knowledge for the coalition members that he would intend to join.

A particular point of the core is defined to be an equilibrium. It is analogous to defining the Walrasian equilibrium in the core of an economy with private goods. Arrow [1] and Debreu [6] extended the notion of Walrasian equilibrium to economies under uncertainty, assuming that the market offers a complete set of state-contingent claims. Under the presence of differential information, the Arrow-Debreu equilibrium distorts the allocation in favor of an agent having superior information, since he can sell to informationally inferior agents a contract for delivery contingent on an event which he knows will not occur. The recent theory of the rational-expectations equilibrium (for example, Grossman [11]) remedies this deficiency by incorporating a signaling mechanism into their model–uninformed agents may be able to infer the information by observing prevailing prices and/or trading volumes. In contrast, our equilibrium notion corrects the distortion by imposing a special type of quantity constraints, which we interpret as the prohibition of "inside trades."

Section 2 provides the formulation of the syndicate problem. In Section 3 we define the concepts of *conditional efficiency* and the *conditional core*, and in Section 4 we define our equilibrium notion, which is called the *constrained competitive equilibrium*. Section 5 provides the basic set of mathematical assump-

3 of this paper. It will play a crucial role in our construction.

tions which is required for subsequent theoretical results. The main result of Section 6 is that an equilibrium contract belongs to the core. Section 7 provides an existence theorem for equilibrium contracts. In Section 8 we give a simple example of the syndicate problem and analyze the nature of the equilibrium contract, assuming that the agents have (1) logarithmic and (2) exponential utility functions.

2 Formulation

The syndicate problem is modeled as a game characterized by (i) a set of states, together with a σ-field of events; (ii) a set of agents, each of whom is described by a probability measure on the state space, risk preferences, and information; (iii) each coalition's opportunity for joint actions.

The State Space

Let Ω be a set, to be interpreted as the set of alternatives states. Associated with Ω is a σ-field of events, or measurable subsets, denoted by $\mathscr{F}$; thus, $(\Omega,\mathscr{F})$ is a measurable space. The occurrence of an event is determined by Nature and is beyond the control of any of the agents. The measurable space $(\Omega,\mathscr{F})$ together with a probability measure on $\mathscr{F}$ defines a probability space.

The information of an agent is described by a sub-σ-field of events that he can discern. He can discern an event E if and only if he knows whether the prevailing state is in the event E or in the complementary event $\Omega \setminus E$. For example, if an agent observes the value of a real-valued random variable $\boldsymbol{y}$ that is measurable with respect to (w.r.t.) $\mathscr{F}$,[4] then the corresponding sub-σ-field, say $\mathscr{G}$, is the field of events that are the inverse images by $\boldsymbol{y}$ of Borel sets in the reals (R). The random variable $\boldsymbol{y}$ induces a partition of Ω such that each member of the partition is a minimal nonempty event in $\mathscr{G}$. Instead of dealing with a general sub-σ-field as information, we assume that any information $\mathscr{G}$ has this property: It contains a partition of Ω whose members are minimal nonempty events in $\mathscr{G}$. We call this partition the *finest partition* of $\mathscr{G}$. Then, precisely one member of the partition is known by the agent to contain the prevailing state. $\mathscr{G}(w)$ denotes the unique member of the partition containing the state w. If the prevailing state is w, an agent having information $\mathscr{G}$ is informed that the prevailing state is in the event $\mathscr{G}(w)$, or, in other words, he observes the event $\mathscr{G}(w)$.

Agents

A *finite* set N denotes the set of all agents. Each agent i has subjective probability judgment on events, which is given by a probability measure μ_i on $(\Omega,\mathscr{F})$. His risk preferences are represented by a real-valued utility function $\boldsymbol{u}_i$, in which

[4] A real-valued function $\boldsymbol{y}$ defined on Ω is said to be measurable w.r.t. $\mathscr{F}$ if $\{w : \boldsymbol{y}(w) \leq y\} \in \mathscr{F}$ for each real number y.

$\boldsymbol{u}_i(x, w)$ is his utility of having x wealth units in state w. We assume that each $\boldsymbol{u}_i(x, \cdot)$ is $\mathscr{F}$-measurable for each x.

We assume that all the agents have common information on the day when the contract is delivered. Let $\mathscr{H}$ denote the sub-σ-field of $\mathscr{F}$ that represents this information. A *wealth share* is an $\mathscr{H}$-measurable random variable. Agent i's utility of a wealth share $\boldsymbol{x}$ is the random variable $\boldsymbol{u}_i[\boldsymbol{x}]$ that is defined by $\boldsymbol{u}_i[\boldsymbol{x}](w) = \boldsymbol{u}_i(\boldsymbol{x}(w), w)$ for each w.

Each agent initially has private information. It is denoted by $\mathscr{G}_i$ where $\mathscr{G}_i$ is contained in the sub-σ-field $\mathscr{H}$. We will later assume that $\mu_i(\mathscr{G}_i(w)) > 0$ for each i and w. If w is the prevailing state, agent i prefers a wealth share $\boldsymbol{x}$ to another $\boldsymbol{x}'$ if and only if $\mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}]|\mathscr{G}_i\}(w) > \mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}']|\mathscr{G}_i\}(w)$.[5] Namely, agent i prefers $\boldsymbol{x}$ and $\boldsymbol{x}'$ at state w iff the conditional expected utility of $\boldsymbol{x}$ (w.r.t. the prior probability measure μ_i) given the information $\mathscr{G}_i$ is larger than that of $\boldsymbol{x}'$. The formulation allows null information. That $\mathscr{G}_i$ is the null field, $\{\phi, \Omega\}$, for every i corresponds to the situation in which there is initially no private information.

Beside the initial information, the agents may obtain additional information from external sources by the time they take a joint action. Let $\mathscr{G}_i'$ denote agent i's information when a joint action is taken. We assume that $H \supset \mathscr{G}_i' \supset \mathscr{G}_i$ for each i ; namely, no one forgets his previous information. We assume without loss of generality that the meet of $\mathscr{G}_i$, denoted by $\wedge_{i\in N}\mathscr{G}_i$, is the null partition;[6] that is, the minimal event that everyone knows to be prevailing is Ω.

Coalitions

Each nonempty subset of N is called a *coalition*. This includes N itself, as well as coalitions of a single agent. The *action space* of a coalition C, denoted by $A(C)$, is the set of all joint actions available for the coalition. A function $\boldsymbol{z}^C : A(C) \times \Omega \to R$ denotes the payoff function of coalition C: If C chooses an action α in $A(C)$ and w is the prevailing state, then the coalition obtains $z^c(\alpha, w)$ wealth units. We assume that $\boldsymbol{z}^C(\alpha, \cdot)$ is $\mathscr{H}$-measurable for each α. It is an important restriction of our model that there be no physical externality among the actions of disjoint coalitions.

A *contract* that a coalition C proposes for its members is described by a *strategy* and a *sharing plan*. A strategy is a function $\boldsymbol{\delta}^C$ from Ω to $A(C)$, which specifies a joint action that coalition C takes in each state. To examine fully the cooperative nature of the problem, we assume that each coalition utilizes the

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[5]Given a probability space $(\Omega, \mathscr{F}, \mu_i)$ and a sub-σ-field $\mathscr{G}_i$, the conditional expectation of an F-measurable random variable $\boldsymbol{y}$ is denoted by $\mathscr{E}_i\{\boldsymbol{y}|\mathscr{G}_i\}$. It is a $\mathscr{G}_i$-measurable random variable which is defined by

$$\mathscr{E}_i\{\boldsymbol{y}|\mathscr{G}_i\}(w) = \int_{\mathscr{G}_i(w)} \boldsymbol{y}(w) d\mu_i(w)/\mu_i(\mathscr{G}_i(w)), \qquad \forall w \in \Omega,$$

under the assumption that $\mathscr{G}_i$ has a finest partition of Ω.

[6]If $\{\mathscr{G}_i\}_{i\in C}$ is a collection of σ-fields of Ω, then $\wedge_{i\in C}\mathscr{G}_i$ denotes their meet: It is the maximal σ-field contained in all of them, and it equals the intersection of $\mathscr{G}_i, i \in C$.

full information available to its members on taking an action. Namely, members of a coalition release their private information honestly and the coalition pools the information *on the date of action*. Thus, $\boldsymbol{\delta}^C$ is required to be measurable w.r.t. the sub-σ-field $V_{i\in C}\mathscr{G}_i'$.[7] [8] The strategy space of a coalition C is denoted by $\Delta(C)$. A sharing plan, denoted by $\boldsymbol{x}^C$, is a collection of wealth shares $\boldsymbol{x}_i^C$,one for each member of coalition C A contract $\langle \boldsymbol{\delta}^C, \boldsymbol{x}^C \rangle$ is said to be *feasible* at a state w for coalition C if and only if it satisfies the inequality:

$$\sum_{i\in C} \boldsymbol{x}_i^C(w) \leq \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w).$$

Dates

There are three dates in our model: the initial date or the date of contracting, the date of action, and the date of contract delivery. Prior to the initial date the agents observe private information $\mathscr{G}_i$ about the prevailing state. At the initial date, they negotiate to reach a contract-an agreement on the course of action to be undertaken (or, a strategy) and an agreement on a provision for allocating the syndicate's wealth (or, a sharing plan). They may obtain additional information by the date of action; the new information of agent i is $\mathscr{G}_i'(\supset \mathscr{G}_i)$. The actual action is taken at the date of action following the strategy specified in the contract and the information currently available to the syndicate. A new information, due to the larger σ-field $\mathscr{H}$, is revealed to everyone at the date of contract delivery. The syndicate's wealth is realized, and each member receives his share of wealth.

3 Conditional Core and Conditional Efficiency

We use the concept of the *core* to define solutions to the syndicate problem. The basic idea is to select those contracts which have the property that if one is proposed no subset of agents has an incentive to opt for an alternative contract. In the terminology of game theory, the core consists of contracts which no coalition of agents can *block*. A coalition of agents can *block* a contract if it can propose an alternative contract which is enforceable by a collective action of the coalition members, and under which every member of the coalition is made better off. The notion of *efficiency* (*Pareto optimality*), which is less stringent a notion, considers possibilities of blocking only by the *grand coalition*, or the coalition consisting of all agents: A contract is efficient (*Pareto optimal*) if it is not blocked by the grand coalition.

It is straightforward to define blocking, the core, and efficiency for games in which players' preferences over a set of possible contracts are fixed (see, for

[7] If $\{\mathscr{G}_i\}_{i\in C}$ is a collection of σ-fields of Ω, then $\vee_{i\in C}\mathscr{G}_i$ denotes their join: It is the minimal σ-field containing all $\mathscr{G}_i, i \in C$.

[8] A function $\boldsymbol{\delta}^C : \Omega \to A(C)$ is said to be measurable w.r.t. a σ-field $\mathscr{G}$ if and only if $\{w : \boldsymbol{\delta}^C(w) = \alpha\} \in \mathscr{G}$ for each $\alpha \in A(C)$.

example, Scarf [17]). However, it is not so straightforward when agents have different information. The difficulty lies in the statement that "every member of the coalition is made better off." To define the content of this statement, one must cope with the prospect of communication; namely, various blocking actions taken by players in negotiation processes may evoke the exchange of information, which may result in shifts of their preferences.

Wilson [22] investigates the issue by considering a number of core notions which differ in the degree to which communication is permitted among players. In this paper we assume that players agree not to exchange their private information at the negotiation table. Agents with superior information may be reluctant to communicate, in anticipation of reaching a favorable agreement by hiding their information. Even if formal communication is prohibited, there still remain possibilities of leakage of information via the announcements of players' intentions to or not to participate in a blocking coalition. They may possess a range of manipulative strategies to control the "amount" of information leakage via a consistent course of action, particularly by incorporating the device of randomization. This is an interesting topic in the theory of noncooperative games (see, for example, Ponssard [13]), but, as such, it is beyond the scope of this paper.

The core whose definition will be given subsequently is identical to the Wilson *coarse core*. We will set forth this concept using Aumann's notion of *common knowledge*. Basically, we intend to eliminate those contracts which, if they are proposed, are likely to be blocked by some coalition without involving any exchange of information among its members.

We start by defining the agents' preferences for contracts. An agent's preference for a contract is solely determined by the wealth share assigned to him, namely:

Definition 1. *An agent i, in a state w, prefers a contract which assigns him a wealth share $\boldsymbol{x}$ to another contract which assigns him a wealth share $\boldsymbol{x}'$ if and only if*

$$\mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}]|\mathscr{G}_i\right\}(w) > \mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}']|\mathscr{G}_i\right\}(w).$$

The next step is to give content to the statement that "every member of a coalition is made better off" by a contract. By this statement we will mean more than just that every member of a coalition prefers one contract to another. The notion of common knowledge plays a key role. To take a coalition of two agents, by the above statement we will require not only that both 1 and 2 prefer one contract to another, but also that 1 knows that 2 prefers one to another, 2 knows that 1 prefers one to another, 1 knows that 2 knows that 1 prefers one to another, and so on. Formally, we define the notion of common knowledge as follows.

Definition 2. *Given a coalition C and a state w, an event E is said to be common knowledge at state w for coalition C if and only if E includes the* *member of the finest partition of the meet* $\wedge_{i \in C} \mathscr{G}_i$ *that contains* w*; namely,*

$$E \supset \left(\wedge_{i \in C} \mathscr{G}_i \right) (w)$$

Definition 3. *A contract is blocked at a state* w *by a coalition* C *if and only if there exists an alternative contract having the following property: It is common knowledge at state* w *for coalition* C *that (i) it is feasible for* C*, and (ii) no member of the coalition prefers the originally proposed contract to the alternative contract, and at least one member prefers the alternative contract.*

Definition 4. *The conditional core is the set of all contracts that have the following property: At each state* w *in* Ω*, (i) they are feasible for* N*; (ii) no coalition can block them. A contract is conditionally efficient if and only if at each state* w *in* Ω*, (i) it is feasible for* N*; (ii) coalition* N *cannot block it.*

To see that the formal definition of common knowledge is equivalent to the informal description, take a coalition of two agents. Suppose that w is the true state and E is an event. To say that 1 knows E at state w means that E includes the observed event $\mathscr{G}_1(w)$ To say that 1 knows at state w that 2 knows E means that, for any state w' in $\mathscr{G}_1(w)$, E includes the event $\mathscr{G}_2(w')$. To say that 1 knows at state w that 2 knows that 1 knows E means that, for any state w' in $\mathscr{G}_1(w)$ and w'' in $\mathscr{G}_2(w')$, E includes the event $\mathscr{G}_1(w'')$. And so on. In general, call a state w' *reachable from* w if there is a sequence of events $E_1, E_2, \ldots, E_k$ such that $w \in E_1, w' \in E_k$, and consecutive E_j intersect and belong to the finest partitions of two different agents. Then all the sentences of the form "i_1 knows at state w that i_2 knows that i_3 knows ... E" (where $i_1 \neq i_2, i_2 \neq i_3, \ldots$) are true if and only if E contains all w' reachable from w. But the set of all w' reachable from w coincides with that member of the finest partition of the meet $\wedge_{i \in C} \mathscr{G}_i$ that contains w; so the desired equivalence is established.[9]

To define the notion of blocking that is not subject to leakage of information, it is central to require that it is common knowledge for every member of a blocking coalition that all of them prefer the counterproposal to the proposed contract–or, to be more precise, that none of them prefer the proposed contract to the counterproposal and at least one of them prefers the counterproposal. To illustrate the point, suppose that Ω has two states w_1, w_2 of equal prior probability and consider a coalition of two risk-averse agents. Agent 1 has the partition $\{\{w_1\}, \{w_2\}\}$ and agent 2 has the null partition $\{w_1, w_2\}$. The prevailing state is w_1 (which 1 knows but 2 does not). A contract is proposed that will provide agent 1 and agent 2 with 0 and 1 wealth units, respectively, if the state is w_1; and 1 and 0 wealth units, respectively, if the state is w_2. Consider the alternative contract that will provide each agent with 1/2 wealth units for certain. Obviously, 1 is better off with the counterproposal and 2 is also better off; but 2 does not know if 1 is better off. Thus, it is not common knowledge that 1 and 2 both prefer the counter-proposal. If both announce

[9]This argument is due to Aumann [4].

their intentions to block the original contract, 2 will immediately learn that the true state must be w_1 and will decline to participate in the blocking coalition.

For another example, suppose that Ω has four states w_1, w_2, w_3, w_4 of equal probability, and consider again a coalition of two risk-averse agents. Let agent 1's partition be

$$\{\{w_1, w_2\}, \{w_3, w_4\}\}$$

and agent 2's partition be

$$\{\{w_1, w_2, w_3\}, \{w_4\}\},$$

and suppose the prevailing state is w_1. A contract is proposed that will provide agent 1 and agent 2 with 1/2 wealth units if w_1 prevails; 0 and 1 wealth units, respectively, if either w_2 or w_4 prevails; and 1 and 0 wealth units, respectively, if w_3 prevails. Consider again the alternative contract that will provide each with 1/2 wealth units for certain. With this counterproposal 1 and 2 know that both of them are better off. But 2 does not know if 1 knows that 2 is better off; so it is not common knowledge that 1 and 2 both prefer the counterproposal. If they announce their favor for the sure contract, it will become common knowledge that the true state is not w_4. Note that although 1 and 2 knew in advance from their private information that the true state is not w_4, it was not common knowledge before the announcements. Now 2 does not know any more whether 1 prefers the sure contract, because if the true state is w_3 then 1 can identify the state by his private information and the original contract will provide him with the maximal 1 wealth unit. Nevertheless, their favor for the counterproposal is still unchanged and hence neither one would turn it down. Then 2 will learn that the true state is not w_3 and that he is worse off with the counterproposal. Thus, although both agents at first favored to block the proposed contract, the sequence of announcements and reasoning lead agent 2 to learn eventually that the true state is either w_1 or w_2, thereby to decline to form the proposed blocking coalition. Even though it may be initially to every member's interest to turn down a proposed contract and opt for another, the agreement among coalition members to block is successfully reached without any leakage of information if and only if it is common knowledge that every member prefers to block.[10]

It is important to note that the conditional core of Definition 4 does not depend on the agents' initial observations. One may propose an alternative definition of the core in which a core contract at a state is such that it is not blocked by any coalition *at that state*, thus associating the core to each prevailing state. The following simple example illustrates the deficiency of this definition.

Consider a syndicate problem of two agents in, which there is no opportunity of actions. Each agent has random wealth endowment, and the syndicate's role

[10] One can construct an example in which the successive leakage of information does not alter members' original intentions to block. For example, consider the situation dealt with in the second example above, and suppose that a contract is proposed that will provide 1 and 2 with 1 and 0 wealth units if either w_1 or w_3 prevails and 0 and 1 wealth units otherwise. Consider the same alternative contract as before. Then 2 will eventually learn that the true state is either w_1 or w_2, but his preference of the sure contract over the original contract will be unaltered. Our definition fails to recognize this possibility of blocking.

is merely to reallocate the wealth for the purpose of mutual insurance. There are two states w_1 and w_2. Assume that the wealth endowment of each agent is the same as that the proposed contract provided him in the first example. Assume that the information of each agent is also the same as before. Then, the sure contract of giving 1/2 wealth units to each agent would be a core contract at state w_1. It is feasible at each state; neither agent 1 nor agent 2 blocks it by himself, and the coalition of 1 and 2 cannot block it either. (For the two-member coalition to block it, they must find another feasible contract which will give agent 1 more than 1/2 wealth units in states w_1 and w_2, because of the common knowledge requirement for blocking. Such a contract will make agent 2 worse off.) This contract is extremely favorable to agent 1. He has no wealth endowed in the prevailing state (and he knows what the prevailing state is!), but he obtains 1/2 wealth units from agent 2. On the other hand, agent 2 knows that if the prevailing state is w_2, agent 1 will block this contract by himself. Hence, by observing that agent 1 did not exert his blocking power, agent 2 knows the true state and declines to accept the sure contract.

To prohibit the information leakage to outside agents via a blocking or non-blocking action of a coalition, we require that a core contract be such that it is common knowledge for everyone that no coalition blocks the contract. But, since $\wedge_{i\in N}\mathscr{G}_i = \{\phi, \mathit{\Omega}\}$ by assumption, this is equivalent to the requirement in Definition 4, namely that no coalition blocks the contract *at each state*. The core of the above example, then, consists only of the initial endowment. Clearly, the conditional core concept is identical to the ordinary core concept when there is no differential information.

4 Constrained Competitive Equilibrium Contracts

The conditional core usually contains a continuum of contracts. Thus, we have the problem of choosing a contract from the core. Unfortunately, no operational method is available to identify all the core contracts. In this section we provide a stronger concept of *constrained competitive equilibria*, which enables us to mathematically identify one or a subset of the core contracts.

Let $\mathit{\Pi}$ denote the set of all probability measures on $(\mathit{\Omega}, \mathscr{F})$, whose element is generically denoted by $\boldsymbol{\pi}$. The expectation of a random variable $\boldsymbol{y}$ on the probability space $(\mathit{\Omega}, \mathscr{F}, \boldsymbol{\pi})$ is denoted by $\mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{y}\}$.

Definition 5. *A budget plan, $\boldsymbol{d}$, is a collection of n functions $\boldsymbol{d}_i : \mathit{\Pi} \times \mathit{\Omega} \to R$, $i \in N$ such that for each $\boldsymbol{\pi} \in \mathit{\Pi}$, (i)$\boldsymbol{d}_i(\boldsymbol{\pi}, \cdot)$ is measurable w.r.t.$\mathscr{G}_i$ and*

(ii)
$$\mathscr{E}_{\boldsymbol{\pi}}\left\{\sum_{i\in N}\boldsymbol{d}_i(\boldsymbol{\pi},\cdot)\right\} = \sup_{\boldsymbol{\delta}^N\in\mathit{\Delta}(N)} \mathscr{E}_{\boldsymbol{\pi}}\left\{\boldsymbol{z}^N(\boldsymbol{\delta}^N(\cdot),\cdot)\right\}.$$

Given a probability measure $\boldsymbol{\pi}$, a budget plan specifies a collection of random budgets, one for each member of the syndicate. The random budget assigned to each member is measurable w.r.t. his initial information; that is, he is given a budget which he can spend to "purchase" a wealth share, depending on the information he has on the initial date. Condition (ii) above requires that the expected sum of individual budgets must equal the syndicate's maximum expected wealth, in which the expectations are taken using the probability measure $\boldsymbol{\pi}$.

Definition 6. *A contract $\langle \bar{\boldsymbol{\delta}}^N, \bar{\boldsymbol{x}}^N \rangle$ is said to be a constrained competitive equilibrium contract relative to a budget plan $\boldsymbol{d} = \{\boldsymbol{d}_i\}_{i \in N}$ iff there exists a probability measure $\bar{\boldsymbol{\pi}}$ in Π such that*

(i) $$\sum_{i \in N} \bar{\boldsymbol{x}}_i^N(w) = \boldsymbol{z}^N(\bar{\boldsymbol{\delta}}^N(w), w), \qquad \forall w \in \Omega.$$

(ii) $$\mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\boldsymbol{z}^N(\bar{\boldsymbol{\delta}}^N(\cdot),\cdot)\right\} \geq \mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\boldsymbol{z}^N(\boldsymbol{\delta}^N(\cdot),\cdot)\right\}, \quad \forall \boldsymbol{\delta}^N \in \Delta(N), \quad \textit{and}$$

(iii)for each i and w

(a) $$\mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\bar{\boldsymbol{x}}_i^N | \mathscr{G}_i\right\} \leq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w), \quad \textit{and}$$

(b) $$\mathscr{E}_i\left\{\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i^N(\cdot),\cdot)|\mathscr{G}_i\right\}(w) \geq \mathscr{E}_i\left\{\boldsymbol{u}_i(\boldsymbol{x}_i(\cdot),\cdot)|\mathscr{G}_i\right\}(w)$$

for any wealth share $\boldsymbol{x}_i$ of agent i satisfying (a).

These conditions can be restated as follows. Condition (i) says that the contract allocates all the syndicate wealth to its members. Condition (ii) says that the equilibrium strategy must attain the syndicate's maximum expected wealth, in which the expectation is taken using the probability measure $\bar{\boldsymbol{\pi}}$. Condition (iii) says that the equilibrium wealth share of each agent is such that (a) its expectation w.r.t. $\bar{\boldsymbol{\pi}}$ conditional on the initial information does not exceed his budget, and (b) the equilibrium wealth share yields his maximum preference among all wealth shares satisfying the budget constraint.

One can interpret the constrained competitive equilibrium contract as the competitive (or Walrasian) outcome of a hypothetical market for state-contingent claims. A claim contingent on a state w entitles its purchaser to one unit of wealth on the date of delivery if the true state is w. For a sharing plan $\boldsymbol{x}^N = \{\boldsymbol{x}_i^N\}_{i \in N}, \boldsymbol{x}_i^N(w)$ is then the number of w-contingent claims that agent i purchases in the market. Given a strategy of the syndicate $\boldsymbol{\delta}^N$, $\boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)$ is the total supply of w-contingent claims. The informational restraint on the date of contract delivery imposes the condition that the "commodity bundles" that can be traded in this market are "bundles" of state-contingent claims which are measurable w.r.t. the then prevailing informational field $\mathscr{H}$. A probability measure $\boldsymbol{\pi}$ in Π is interpreted as a price system; namely, if $\boldsymbol{\pi}$ is the prevailing price system, the market value of a wealth share $\boldsymbol{x}$ is given by $\int_\Omega \boldsymbol{x}(w) d\boldsymbol{\pi}(w) \equiv \mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{x}\}$.[11]

The peculiarity of this hypothetical market lies in the definition of "consumers" in this market. Each agent is split into a number of consumers, who act independently having different amounts of budgets and different preferences. Each consumer who is generated by agent i is associated with a member of the finest partition of $\mathscr{G}_i$; thus, a consumer is represented by a pair (i, E_i),

[11] Later, we will restrict $\boldsymbol{\pi}$ to the measurable subspace $(\Omega, \mathscr{H})$ and require that $\boldsymbol{\pi}(E) = 0$ for each $E \in \mathscr{H}$ such that $\mu_i(E) = 0, \forall_i$.

in which $i \in N$ and $E_i = \mathscr{G}_i(w)$ for some w. The preferences of consumer (i, E_i) are defined in such a way that his preference for a wealth share $\boldsymbol{x}$ is given by the conditional expected utility $\int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w) d\mu_i(w)/\mu_i(E_i)$. For the technical convenience of avoiding insatiable preferences, one may restrict consumption sets of the consumers such that consumer (i, E_i) may not purchase any claim contingent on a state outside E_i. There is only one "producer" in this market, whose technological possibilities are defined by the strategy space $\Delta(N)$ and the payoff function $\boldsymbol{z}^N$ of the syndicate. A strategy $\boldsymbol{\delta}^N$ generates supply of state-contingent claims, whose market value (or profit) is given by $\int_{\Omega} \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w) d\boldsymbol{\pi}(w) \equiv \mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{z}^N(\boldsymbol{\delta}^N(\cdot), \cdot)\}$ if the prevailing price system is $\boldsymbol{\pi}$. This profit is distributed among the consumers according to a budget plan $\boldsymbol{d}$.[12]

Then the conditions defining a constrained competitive equilibrium are simply that the producer maximizes his profit, that each consumer purchases the most preferred commodity bundle subject to his budget constraint, and that total supply equals total demand, which is the standard requirement for a competitive equilibrium. Note that if each consumer exhausts his budget (*i.e.*,

$$\mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{x}_i | \mathscr{G}_i\}(w) = \boldsymbol{d}_i(\boldsymbol{\pi}, w))$$

and if $\boldsymbol{\delta}^N$ maximizes profit at $\boldsymbol{\pi}$, then condition (ii) of Definition 5 yields

$$\int_{\Omega} \left[\sum_{i \in N} \boldsymbol{x}_i(w) - \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w) \right] d\boldsymbol{\pi}(w) = 0,$$

which, is the Walras' law.

The peculiar feature of the concept of constrained competitive equilibria is the prohibition of "inside trades." An agent i's choice at a state w is restricted to trading claims contingent on states which belong to the prevailing event $\mathscr{G}_i(w)$. He is not allowed to generate income by selling short the claims contingent on a state which he knows will never occur. This is in constrast to the *Arrow-Debreu equilibria* which are constructed on the basis of agents' posterior preferences. As Radner [15] states:

> If an agent knew that an event E obtained, but not all agents knew this, the agent in question might find himself in a position in which he could sell, at a positive price, a contract for delivery contingent on an event that he already knew could not occur. *Caveat emptor* ! Whether or not this raises any moral questions, it does raise the question of whether or not an agent's information includes knowledge of other agents' information structures. In the real world of

[12] For this interpretation, express the budget constraint (a) of condition (iii) as

$$\int_{\mathscr{G}_i(w)} \bar{\boldsymbol{x}}^N(w') d\bar{\boldsymbol{\pi}}(w') \leq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w) \cdot \bar{\boldsymbol{\pi}}(\mathscr{G}_i(w))$$

and regard the quantity on the right-hand side as the profit distributed to the consumer $(i, \mathscr{G}_i(w))$.

> contracts between individuals, this question does arise. For example, it is not considered correct to make a bet on the outcome of a race whose results you already know. But in the Arrow-Debreu world, individuals make contracts with an impersonal "market", so this issue does not arise, and individuals are free to make such contracts.

The prohibition of "inside trades" distinguishes our equilibrium concept from the concept of the Arrow-Debreu equilibria and motivates the name constrained competitive equilibria. Of course, if there is no differential information, then the two equilibrium concepts coincide.

If $\boldsymbol{\pi}$ is the relevant probability measure (price system) to compute the expected wealth (profit) and w is the prevailing state, then a coalition C can claim a budget of at least the amount

$$\sup_{\boldsymbol{\delta}^C \in \Delta(C)} \mathscr{E}_{\boldsymbol{\pi}} \left\{ \boldsymbol{z}^C(\boldsymbol{\delta}^C(\cdot), \cdot) \,\middle|\, \bigwedge_{i \in C} \mathscr{G}_i \right\} (w),$$

since the coalition can obtain this amount by acting independently using its opportunity for joint actions. The following property of a budget plan is necessary to ensure that a constrained competitive equilibrium contract is a core contract.

Definition 7. *A budget plan* $\boldsymbol{d} = \{\boldsymbol{d}_i\}_{i \in N}$ *is said to be core-compatible iff it satisfies, for each coalition* C *and for each* $\boldsymbol{\pi} \in \Pi$*, the inequality:*

$$\begin{aligned} &\mathscr{E}_{\boldsymbol{\pi}} \left\{ \sum_{i \in C} \boldsymbol{d}_i(\boldsymbol{\pi}, \cdot) \,\middle|\, \bigwedge_{i \in C} \mathscr{E}_i \right\} (w) \\ &\quad \geq \sup_{\boldsymbol{\delta} \in \Delta(C)} \mathscr{G}_{\boldsymbol{\pi}} \left\{ \boldsymbol{z}^C(\boldsymbol{\delta}^C(\cdot), \cdot) \,\middle|\, \bigwedge_{i \in C} \mathscr{G}_i \right\} (w), \qquad \forall w \in \Omega. \end{aligned}$$

In Section 6 we will prove that constrained competitive equilibrium contracts are in the conditional core if the budget plan is chosen to be core-compatible. But nothing thus far indicates that the equilibrium solution plays a privileged role over other core contracts. Debreu and Scarf [7] show for an exchange economy consisting of a finite number of types of consumers that the core shrinks to the set of competitive equilibria as the number of each type becomes infinite. They also extend the theorem to the case of a productive economy with constant-returns-to-scale technology. Instead of "replicating" an economy, Aumann [3] considers an economy with a continuum of traders and shows that the core and the set of competitive equilibria coincide. It seems natural to conjecture that a similar theorem is available for our syndicate problem. If the syndicate is a large organization consisting of a continuum of agents so that every agent is "insignificant" relative to the total, the constrained competitive equilibrium contracts would be the only solutions to the problem that are free from coalition oppositions. On the other hand, if the size of the syndicate is small it is difficult to favor the equilibrium contracts over others on sound ethical ground. However, the equilibrium contracts are amenable to mathematical analysis and have appealing properties, as will be exhibited in Section 8.

5 Basic Assumptions

We earlier defined a wealth share to be a real-valued random variable measurable w.r.t. the sub-σ-field $\mathscr{H}$, in which $\mathscr{H}$ is the information available to everyone on the day of contract delivery. But, the set of all $\mathscr{H}$-measurable random variables is too large to be mathematically manageable. Thus, we restrict the space of wealth shares in the following way.

Given the individual probability measures μ_i on $(\Omega, \mathscr{F})$, define a new probability measure μ by

$$\mu(E) = \frac{1}{n} \sum_{i \in N} \mu_i(E), \qquad \forall E \in \mathscr{F},$$

in which n is the number of agents in N. As we only deal with $\mathscr{H}$-measurable wealth shares, we restrict to μ the measurable space$(\Omega, \mathscr{H})$. For a random variable $\boldsymbol{x}$ on $(\Omega, \mathscr{H}, \mu)$ define the *essential supremum* by

$$\operatorname{ess\,sup} |\boldsymbol{x}(w)| = \inf_{E} \sup_{w \in \Omega \setminus E} |\boldsymbol{x}(w)|,$$

where E ranges over the μ-null subsets of Ω. Let $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ denote the space of all essentially bounded, real-valued, $\mathscr{H}$-measurable functions on Ω, considering two functions in $\mathscr{L}_\infty$ to be equivalent iff they are equal almost everywhere. Namely,

$$\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu) = \{\boldsymbol{x} : \Omega \to R | \boldsymbol{x} \text{ is } \mathscr{H}\text{-measurable and ess sup} \, |\boldsymbol{x}(w)| < \infty \}.$$

With the norm

$$\|x\|_\infty = \operatorname{ess\,sup} |\boldsymbol{x}(w)|,$$

$\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ is a normed linear space. We will take the wealth shares in this space.

We impose the following assumptions on each agent's set of wealth shares and his utility function.

Assumption 1. *For each i, (i) the set of wealth shares of agent i is*

$$\mathscr{L}_\infty^{b_i}(\Omega, \mathscr{H}, \mu) \equiv \mathscr{L}_\infty(\Omega, \mathscr{H}, \mu) \cap \{\boldsymbol{x} : \Omega \to R | \boldsymbol{x}(w) \geq b_i \ a.e.\},$$

in which b_i is a nonnegative number. The utility function $\boldsymbol{u}_i$ is a function from $[b_i, \infty) \times \Omega$ to R such that (ii) for each w, $\boldsymbol{u}_i(x, w)$ is a continuous, concave, and strictly increasing function of x such that $\boldsymbol{u}_i(b_i, w) = 0$, and (iii) for each x, $\boldsymbol{u}_i(x, \cdot)$ is integrable on $(\Omega, \mathscr{F}, \mu_i)$.

Proposition 1. *Under Assumption 1, if $\boldsymbol{x}$ is in $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$, then the function $\boldsymbol{u}_i[\boldsymbol{x}] : \Omega \to R$, defined by $\boldsymbol{u}_i[\boldsymbol{x}](w) = \boldsymbol{u}_i(\boldsymbol{x}(w), w), \forall w \in \Omega$, is measurable w.r.t.$\mathscr{F}$ and $\mathscr{E}_i \{\boldsymbol{u}_i[\boldsymbol{x}] | \mathscr{G}_i\}$ is finite-valued for each i.*

Proof: Let $\boldsymbol{x} \in \mathscr{L}_\infty$. To show that $\boldsymbol{u}_i[\boldsymbol{x}]$ is measurable w.r.t. $\mathscr{F}$, define $\boldsymbol{x}^K$ for each positive integer K by

$$\boldsymbol{x}^K(w) = \frac{k}{K} \quad \text{if} \quad \frac{k}{K} \le \boldsymbol{x}(w) < \frac{k+1}{K}, \qquad -\infty < k < \infty.$$

For each real number u, we have

$$\{w : \boldsymbol{u}_i(\boldsymbol{x}^K(w), w) \le u\} = \bigcup_{k=-\infty}^{\infty} \left\{w : \boldsymbol{x}^K(w) = \frac{k}{K}\right\} \cap \left\{w : \boldsymbol{u}_i\left(\frac{k}{K}, w\right) \le u\right\}.$$

The set $\{w : \boldsymbol{x}^K(w) = k/K\}$ is in $\mathscr{H}$ and the set $\{w : \boldsymbol{u}_i(k/K, w) \le u\}$ is in $\mathscr{F}$, so the set $\{w : \boldsymbol{u}_i(\boldsymbol{x}^K(w), w) \le u\}$ is in $\mathscr{F}$; hence $\boldsymbol{u}_i[\boldsymbol{x}^K]$ is $\mathscr{F}$-measurable. Since $\boldsymbol{x}^K(w) \to \boldsymbol{x}(w)$, $\forall w$, as $K \to \infty$ and $\boldsymbol{u}_i(x, w)$ is continuous in x, $\boldsymbol{u}_i(\boldsymbol{x}^K(w), w) \to \boldsymbol{u}_i(\boldsymbol{x}(w), w)$, $\forall w$. Namely, $\boldsymbol{u}_i[\boldsymbol{x}]$ is the (pointwise) limit of the sequence of $\mathscr{F}$-measurable functions $\boldsymbol{u}_i[\boldsymbol{x}^K]$. Hence, $\boldsymbol{u}_i[\boldsymbol{x}]$ is $\mathscr{F}$-measurable. By the monotonocity of $\boldsymbol{u}_i(\boldsymbol{x}, w)$ in x we have

$$|\boldsymbol{u}_i(\boldsymbol{x}(w), w)| \le |\boldsymbol{u}_i(\|\boldsymbol{x}\|_\infty, w)| \quad \text{a.e.},$$

but μ-null events are μ_i-null so that the same inequality holds almost everywhere w.r.t. μ_i. Then, since $\boldsymbol{u}_i[\boldsymbol{x}]$ is $\mathscr{F}$-measurable, (iii) implies $\int |\boldsymbol{u}_i[\boldsymbol{x}]| d\mu_i < \infty$. Therefore, $\mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}] | \mathscr{G}_i\}$ is finite-valued. ■

We impose the following assumption on the action opportunity of each coalition.

Assumption 2. *For each coalition C, (i) the action space $A(C)$ is a nonempty, compact, convex, and separable subset of a normed linear space with norm $\|\cdot\|_A^C$. The payoff function $\boldsymbol{z}^C : A(C) \times \Omega \to R$ satisfies: (ii) for each w, $\boldsymbol{z}^C(\alpha, w)$ is a continuous and concave function of α, (iii) for each α, $\boldsymbol{z}^C(\alpha, \cdot)$ is in $\mathscr{L}(\Omega, \mathscr{H}, \mu)$, and (iv) if $\boldsymbol{\delta}^* : \Omega \to A(C)$ is an $\mathscr{H}$-measurable function satisfying*

$$\boldsymbol{\delta}^*(w) \in arg \max_{\alpha \in A(C)} \boldsymbol{z}^C(\alpha, w), \qquad \forall w \in \Omega,$$

then we have ess $\sup |\boldsymbol{z}^C(\boldsymbol{\delta}^*(w), w)| < \infty$.[13]

Noting that the strategy space $\Delta(C)$ of coalition C is the set of all functions $\boldsymbol{\delta}^C : \Omega \to A(C)$ measurable w.r.t. $V_{i \in C} \mathscr{G}_i'$ we obtain the following proposition.

Proposition 2. *Under Assumption 2, if $\boldsymbol{\delta}^C$ is in $\Delta(C)$, then the function $\boldsymbol{z}[\boldsymbol{\delta}^C] : \Omega \to R$, defined by $\boldsymbol{z}^C[\boldsymbol{\delta}^C](w) = \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w), \forall w \in \Omega$, is in $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$.*

Proof: Since the set $A(C)$ is separable, it contains a countable dense subset. Denote this subset by $A_d(C) = \{\alpha_k; k = 1, 2, \ldots\}$. Given $\boldsymbol{\delta}^C \in \Delta(C)$, define $\boldsymbol{\delta}_K^C : \Omega \to A(C)$ for each positive integer K by

$$\boldsymbol{\delta}_K^C(w) \in \left\{\alpha \in A_d(C) : \|\alpha - \boldsymbol{\delta}(w)\|_A^C \le \frac{1}{K}\right\}, \qquad \forall w \in \Omega,$$

[13] The essential supremum of $|\boldsymbol{z}^C(\boldsymbol{\delta}^*(w), w)|$ is well defined, since $\boldsymbol{z}^C(\boldsymbol{\delta}^*(\cdot), \cdot)$ is $\mathscr{H}$-measurable due to the following Proposition 2.

in such a way that $\boldsymbol{\delta}_K^C$ is measurable w.r.t. $V_{i\in C}\mathscr{G}_i^{'}$. For each real number z, we have

$$\left\{w : \boldsymbol{z}^C(\boldsymbol{\delta}_K^C(w), w) \leq z\right\} = \bigcup_{k=1}^{\infty}\left\{w : \boldsymbol{\delta}_K^C(w) = \alpha_k\right\} \cap \left\{w : z^C(\alpha_k, w) \leq z\right\}.$$

The set $\{w : \boldsymbol{\delta}_K^C(w) = \alpha_k\}$ is in $\vee_{i\in C}\mathscr{G}_i^{'}$ and the set $\{w : \boldsymbol{z}^C(\alpha_k, w) \leq z\}$ is in $\mathscr{H}$, so the set $\{w : \boldsymbol{z}^C(\boldsymbol{\delta}_K^C(w), w) \leq z\}$ is in $\mathscr{H}$; hence, $\boldsymbol{z}^C[\boldsymbol{\delta}_K^C]$ is $\mathscr{H}$-measurable. Since $\boldsymbol{\delta}_K^C(w) \to \boldsymbol{\delta}^C(w), \forall w$, as $K \to \infty$ and $\boldsymbol{z}^C(\alpha, w)$ is continuous in α, $\boldsymbol{z}^C(\boldsymbol{\delta}_K^C(w), w) \to \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w), \forall w$. This implies that $\boldsymbol{z}^C[\boldsymbol{\delta}^C]$ is $\mathscr{H}$-measurable. By condition (iv) we have

$$\text{ess sup}\,|\boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w)| \leq \text{ess sup}\,|\boldsymbol{z}^C(\boldsymbol{\delta}^*(w), w)| < \infty$$

and so $\boldsymbol{z}^C[\boldsymbol{\delta}^C]$ is in $\mathscr{L}_\infty$. ■

A contract that a coalition C proposes for its members is a pair $\langle \boldsymbol{x}^C = \{\boldsymbol{x}_i^C\}_{i\in C}, \boldsymbol{\delta}^C\rangle$, in which $\boldsymbol{x}_i^C \in \mathscr{L}_\infty^{b_i}(\Omega, \mathscr{H}, \mu)$, $\forall i$ and $\boldsymbol{\delta}^C \in \Delta(C)$. Proposition 1 ensures the finiteness of the conditional expected utility $\mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}_i^C]|\mathscr{G}_i\}$. Proposition 2 ensures that the coalition's payoff $\boldsymbol{z}^C[\boldsymbol{\delta}^C]$ is in the space $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$. Then, Condition (i) of the definition of constrained competitive equilibria (Definition 6) can be replaced by:

$$\sum_{i\in C} \bar{\boldsymbol{x}}_i^N(w) = \boldsymbol{z}^N(\bar{\boldsymbol{\delta}}^N(w), w) \qquad \text{a.e.} \qquad (\text{w.r.t. } \mu)$$

The *norm dual* of $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ is $ba(\Omega, \mathscr{H}, \mu)$ under the pairing $\boldsymbol{\pi} \cdot \boldsymbol{f} = \int_\Omega \boldsymbol{f}(w)d\boldsymbol{\pi}(w) \equiv \mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{f}\}, \forall f \in \mathscr{L}_\infty$ and $\forall \boldsymbol{\pi} \in ba$, where $ba(\Omega, \mathscr{H}, \mu)$ is the normed linear space of all bounded additive set functions on $(\Omega, \mathscr{H})$ *absolutely continuous* w.r.t. μ with the norm $\|\cdot\|_{ba}$ being

$$\|\boldsymbol{\pi}\|_{ba} = \sup\left\{\sum_{i=1}^{k} |\boldsymbol{\pi}(E_i)| : E_1, \ldots, E_k \text{ is a finite sequence of disjoint sets in } \mathscr{H}\right\}.$$ [14] [15]

We earlier assumed the equilibrium $\boldsymbol{\pi}$ to be a probability measure on $(\Omega, \mathscr{F})$. But, since the wealth shares and the payoffs are in $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$, it is more natural to take $\boldsymbol{\pi}$ in the norm dual $ba(\Omega, \mathscr{H}, \mu)$, which ensures that $\mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{f}\}$ is finite for each $\boldsymbol{f} \in \mathscr{L}_\infty$. That $\boldsymbol{\pi}$ is a probability measure requires, in addition, that $\boldsymbol{\pi}$ be a countably additive set function. Thus, redefine the set Π to be

[14] See, for example, Dunford and Schwarz [8, p. 296]. The *norm dual* of $\mathscr{L}_\infty$ is defined as the set of all continuous (i.e., bounded) linear functionals on $\mathscr{L}_\infty$ w.r.t. the topology on $\mathscr{L}_\infty$ induced by $\|\cdot\|_\infty$

[15] A set function $\boldsymbol{\pi}$ on $(\Omega, \mathscr{H})$ is said to be *absolutely continuous* w.r.t. μ iff $E \in \mathscr{H}$ and $\mu(E) = 0$ implies that $\boldsymbol{\pi}(E) = 0$.

the set of all (countably additive) probability measures on $(\Omega, \mathscr{H})$ absolutely continuous w.r.t. μ.

Since $\mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{f}\}$ is finite for each $\boldsymbol{f} \in \mathscr{L}_\infty$, the conditional expectation of an agent's wealth share w.r.t. $\boldsymbol{\pi}$, which appears in Definition 6, is finite. Similarly, Proposition 2 ensures that the (conditional) expectation w.r.t. $\boldsymbol{\pi}$ of each coalition's random payoff associated with any strategy, in Definitions 5, 6, and 7, is finite.

Assumption 3. *For each i, the initial information $\mathscr{G}_i$ is generated by a finite partition of Ω.*[16] *The measure μ_i assigns a positive probability to each nonempty event in $\mathscr{G}_i$.*

For each coalition C define the norm $\|\cdot\|_\Delta^C$ on its strategy space by

$$\|\boldsymbol{\delta}\|_\Delta^C = \text{ess sup} \|\boldsymbol{\delta}(w)\|_A^C, \qquad \forall \boldsymbol{\delta} \in \Delta(C).$$

The following proposition ensures that the maximum expected payoff of each coalition, in Definitions 5 and 7, exists and is finite.

Proposition 3. *Under Assumptions 2 and 3, the strategy space $\Delta(C)$ of each coalition C is compact w.r.t. $\|\cdot\|_\Delta^C$, and the mapping $\int \boldsymbol{z}^C[\cdot]d\boldsymbol{\pi} : \Delta(C) \to R$, defined by $\int \boldsymbol{z}^C[\boldsymbol{\delta}]d\boldsymbol{\pi} = \int \boldsymbol{z}^C(\boldsymbol{\delta}(w), w)d\boldsymbol{\pi}(w)$, is continuous w.r.t. the topology on $\Delta(C)$ induced by $\|\cdot\|_\Delta^C$ for each $\boldsymbol{\pi} \in \Pi$.*

Proof: Since the action space $A(C)$ of each coalition C is compact, Assumption 3 implies that $\Delta(C)$ is compact. By conditions (ii) and (iv) of Assumption 2 we can apply the dominated convergence theorem to show that the mapping $\int \boldsymbol{z}^C[\cdot]d\boldsymbol{\pi}$ is continuous. ■

The final basic assumption involves the notion of balanced collection of coalitions, which originates in Shapley [18]. A collection of coalitions $\mathscr{B} = \{C\}$ is said to be a *balanced collection* iff there exists nonnegative weights t_C, for each C in $\mathscr{B}$, such that $\sum_{C \in \mathscr{B}, C \ni i} t_C = 1$ for each i in N.

Assumption 4. *The action opportunities are balanced in the following sense: If $\mathscr{B}$ is a balanced collection of coalitions with weights t_C and $\alpha^C \in A(C)$ for each $C \in \mathscr{B}$, then there exists $\alpha_N \in A(N)$ such that $\boldsymbol{z}^N(\alpha^N, w) \geq \sum_{C \in \mathscr{B}} t_C \boldsymbol{z}^C(\alpha^C, w)$ a.e.*

This assumption can be interpreted as follows. The weight $t_C (\leq 1)$ of a coalition C is the amount of time for which the coalition is formed. Each agent is endowed with one unit of time. If a collection of coalitions $\mathscr{B} = \{C\}$ is a balanced collection, each agent can successfully split his endowed time and participate in each coalition C which assumes his membership. The agents also have the option of collectively forming the syndicate, or the coalition N, for one unit of time. Then, Assumption 4 says that they can collectively achieve a

[16] Given any nonempty collection $\mathscr{C}$ of subsets of Ω, the minimal σ-field containing $\mathscr{C}$ is said to be *generated by* $\mathscr{C}$.

higher wealth in each state by participating in the syndicate than by forming each coalition C in $\mathscr{B}$ for t_C units of time, regardless of each coalition's joint action. As a special case, if $\mathscr{B}$ is the collection of disjoint coalitions whose union is N, then the weight of each coalition is one, and the assumption implies that it is to the advantage of disjoint coalitions to combine. Thus, Assumption 4 extends the familiar notion of *superadditivity* to the class of balanced collections of coalitions.

6 Core-compatibility of the Constrained Competitive Equilibrium

We first establish the existence of a core-compatible budget plan.

Theorem 1. *Under Assumptions* 2, 3, *and* 4, *there exists a core-compatible budget plan.*

Proof: Let M denote the set of pairs (i, E_i), in which $i \in N$ and E_i is any member of the finest partition of $\mathscr{G}_i$. For each subset C of N and each member Q of the finest partition of $\wedge_{i\in C}\mathscr{G}_i$, let $[C, Q]$ be the subset of M defined by

$$[C, Q] = \{(i, E_i) \in M : i \in C, E_i \subset Q\}\,.$$

Let $\mathscr{A}$ denote the collection of all these subsets $[C, Q]$. Let $\boldsymbol{\pi}$ be given in Π. Since $\boldsymbol{d}_i(\boldsymbol{\pi}, \cdot)$ is measurable w.r.t. $\mathscr{G}_i$ by definition, it is a collection of scalars, one for each member of the finest partition of $\mathscr{G}_i$. The value $\boldsymbol{d}_i(\boldsymbol{\pi}, w)$ is denoted by $d_i(E_i)$ if $w \in E_i$. Then the problem is to find the numbers $d_i(E_i)$, $(i, E_i) \in M$,that satisfy

$$\sum_{(i,E_i)\in M} d_i(E_i)\mu_i(E_i) = \sup_{\boldsymbol{\delta}^N\in\Delta(N)} \int_\Omega \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)d\boldsymbol{\pi}(w),$$

and

$$\sum_{(i,E_i)\in[C,Q]} d_i(E_i)\mu_i(E_i) \quad\geq\quad \sup_{\boldsymbol{\delta}^C\in\Delta(C)} \int_Q \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w)d\boldsymbol{\pi}(w), \qquad \forall[C, Q] \in \mathscr{A}.$$

By the Shapley-Bondareva theorem (Shapley [18]), this system of inequalities has a solution if and only if

$$\sup_{\boldsymbol{\delta}^N\in\Delta(N)} \int_\Omega \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)d\boldsymbol{\pi}(w)$$
$$\geq \sum_{[C,Q]\in\mathscr{D}} t_{[C,Q]} \sup_{\boldsymbol{\delta}^C\in\Delta(C)} \int_Q \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w)d\boldsymbol{\pi}(w)$$

for any balanced collection $\mathscr{D}$ of subsets in $\mathscr{A}$ with weights $t_{[C,Q]}$. To show that this inequality holds, let $\boldsymbol{\chi}_Q$ denote the indicator function of the set Q. Then, if $\boldsymbol{\delta}^{[C,Q]} \in \Delta(C)$ for each $[CQ] \in \mathscr{D}$, we have

$$\begin{aligned}
&\sum_{[C,Q]\in\mathscr{D}} t_{[C,Q]} \int_Q \boldsymbol{z}^C(\boldsymbol{\delta}^{[C,Q]}(w), w) d\boldsymbol{\pi}(w) \\
&= \int_\Omega \sum_{[C,Q]\in\mathscr{D}} t_{[C,Q]} \boldsymbol{\chi}_Q(w) \boldsymbol{z}^C(\boldsymbol{\delta}^{[C,Q]}(w), w) d\boldsymbol{\pi}(w) \\
&= \int_\Omega \sum_{\substack{[C,Q]\in\mathscr{D} \\ Q\ni w}} t_{[C,Q]} \boldsymbol{z}^C(\boldsymbol{\delta}^{[C,Q]}(w), w) d\boldsymbol{\pi}(w) \\
&\leq \int_\Omega \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w) d\boldsymbol{\pi}(w),
\end{aligned}$$

in which the $\boldsymbol{\delta}^N$ in the final expression is the strategy of N whose existence is guaranteed by Assumption 4. Hence the theorem follows. ■

Theorem 2. *Under Assumptions* 1, 2, *and* 3, *a constrained competitive equilibrium contract relative to a core-compatible budget plan is in the conditional core.*

Proof: Suppose that a constrained competitive equilibrium contract $\langle \bar{\boldsymbol{\delta}}^N, \bar{\boldsymbol{x}}^N \rangle$ relative to a core-compatible budget plan $\boldsymbol{d}$ is blocked at some state w^0 by a coalition C. By the definition of blocking, there exists a contract $\langle \boldsymbol{\delta}^C, \boldsymbol{x}^C \rangle$ for coalition C such that

$$\sum_{i\in C} \boldsymbol{x}_i^C(w) \leq \boldsymbol{z}^C(\boldsymbol{\delta}^C(w), w) \qquad \text{a.e. on} \qquad \left(\underset{i\in C}{\wedge} \mathscr{G}_i \right)(w^0),$$

and

$$\mathscr{E}_i \left\{ \boldsymbol{u}_i[\boldsymbol{x}_i^C] | \mathscr{G}_i \right\}(w) \geq \mathscr{E}_i \left\{ \boldsymbol{u}_i[\bar{\boldsymbol{x}}_i^C] | \mathscr{G}_i \right\}(w), \qquad \forall w \in \left(\underset{i\in C}{\wedge} \mathscr{G}_i \right)(w^0), \qquad \forall i \in C,$$

in which the last inequality is a strict inequality for some $i \in C$. Let $\bar{\boldsymbol{\pi}} \in \Pi$ be the equilibrium probability measure. Then, a strict inequality of the conditional expected utilities implies

$$\mathscr{E}_{\bar{\boldsymbol{\pi}}} \left\{ \boldsymbol{x}_i^C | \mathscr{G}_i \right\}(w) > \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w).$$

Similarly, an equality of the conditional expected utilities implies

$$\mathscr{E}_{\bar{\boldsymbol{\pi}}} \left\{ \boldsymbol{x}_i^C | \mathscr{G}_i \right\}(w) \geq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w),$$

by the following reason. Define i's wealth share $\boldsymbol{x}'_i \in \mathscr{L}_\infty^{b_i}$ by $\boldsymbol{x}'_i(w) = \boldsymbol{x}_i^C(w) + x$, $\forall w \in \Omega$, for an arbitrary positive scalar x. Since $\boldsymbol{u}_i(x, w)$ is strictly increasing in x, we have $\mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}_i']|\mathscr{G}_i\}(w) > \mathscr{E}_i\{\boldsymbol{u}_i[\boldsymbol{x}_i^C]|\mathscr{G}_i\}(w)$. Consider any wealth share $\boldsymbol{x}_i^t$ defined by $\boldsymbol{x}_i^t(w) = t\boldsymbol{x}_i'(w) + (1-t)\boldsymbol{x}_i^C(w), \forall w \in \Omega$ and $0 < t \leq 1$. By the concavity of $\boldsymbol{u}_i(x, w)$ in x, we get

$$\begin{aligned}\mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}_i^t]|\mathscr{G}_i\right\}(w) &\geq t\mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}_i']|\mathscr{G}_i\right\}(w) + (1-t)\mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}_i^C]|\mathscr{G}_i\right\}(w)\\ &> \mathscr{E}_i\left\{\boldsymbol{u}_i[\boldsymbol{x}_i^C]|\mathscr{G}_i\right\}(w).\end{aligned}$$

This implies $\mathscr{E}_{\bar{\boldsymbol{\pi}}}\{\boldsymbol{x}_i^t|\mathscr{G}_i\}(w) \geq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w)$. By letting $t \to 0$, we obtain $\mathscr{E}_{\bar{\boldsymbol{\pi}}}\{\boldsymbol{x}_i^C|\mathscr{G}_i\}(w) \geq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w)$ due to the dominated convergence theorem. Thus, we obtain $\mathscr{E}_{\bar{\boldsymbol{\pi}}}\{\boldsymbol{x}_i^C|\mathscr{G}_i\}(w) \geq \boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, w)$, $\forall i \in C$ and $\forall w \in (\wedge_{i\in C}\mathscr{G}_i)(w^0)$, with at least one strict inequality. Taking the expectation of each side conditional on the event $(\wedge_{i\in C}\mathscr{G}_i)(w^0)$ and summing over all i in C yields

$$\mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\sum_{i\in C}\boldsymbol{x}_i^C\middle|\bigwedge_{i\in C}\mathscr{G}_i\right\}(w^0) > \mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\sum_{i\in C}\boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, \cdot)\middle|\bigwedge_{i\in C}\mathscr{G}_i\right\}(w^0).$$

But, by feasibility of the contract $\langle\boldsymbol{\delta}^C, \boldsymbol{x}^C\rangle$ the first term is less than or equal to $\mathscr{E}_{\bar{\boldsymbol{\pi}}}\{\boldsymbol{z}^C[\boldsymbol{\delta}^C]|\wedge_{i\in C}\mathscr{G}_i\}(w^0)$; hence,

$$\mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\boldsymbol{z}^C[\boldsymbol{\delta}^C]\middle|\bigwedge_{i\in C}\mathscr{G}_i\right\}(w^0) > \mathscr{E}_{\bar{\boldsymbol{\pi}}}\left\{\sum_{i\in C}\boldsymbol{d}_i(\bar{\boldsymbol{\pi}}, \cdot)\middle|\bigwedge_{i\in C}\mathscr{G}_i\right\}(w^0).$$

This contradicts the assumption that $\boldsymbol{d}$ is core-compatible. ■

7 Existence of Constrained Competitive Equilibrium Contracts

As we stated earlier, a constrained competitive equilibrium is the Walrasian (or Arrow-Debreu) equilibrium of the aforementioned hypothetical market for state-contingent claims. Each consumer is represented by a pair (i, E_i), in which $i \in N$ and E_i is a member of the finest partition of $\mathscr{G}_i$. The consumption set of consumer (i, E_i) is a subset of $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$, defined by

$$X_{(i,E_i)} = \bigcup_{\boldsymbol{x}'\in\mathscr{L}_\infty^{b_i}}\left\{\boldsymbol{x} : \boldsymbol{x}(w) = \boldsymbol{x}'(w)\boldsymbol{\chi}_{E_i}(w) \text{ a.e.}\right\}.$$

His preference for a commodity $\boldsymbol{x} \in X_{(i,E_i)}$ is represented by $\int_{E_i}\boldsymbol{u}_i(\boldsymbol{x}(w), w)d\mu_i(w)/\mu_i(E_i)$. He is endowed with a budget $\boldsymbol{d}_i(\boldsymbol{\pi}, w^0)$ with $w^0 \in E_i$, depending on the price system $\boldsymbol{\pi}$. Given a price system $\boldsymbol{\pi}$, his budget set is

$$\left\{\boldsymbol{x} \in X_{(i,E_i)} : \int_\Omega \boldsymbol{x}(w)d\boldsymbol{\pi}(w) \leq d_i(\boldsymbol{\pi}, w^0)\boldsymbol{\pi}(\mathscr{G}_i(w^0))\right\}.$$

The producer, i.e., the syndicate itself, is represented by a production possibility set Y, which is a subset of $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ defined by

$$Y = \bigcup_{\boldsymbol{\delta}^N \in \Delta(N)} \left\{ \boldsymbol{y} \in \mathscr{L}_\infty : \boldsymbol{y}(w) \le \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)\text{a.e.} \right\}.$$

We use Bewley's theorem [5] to show the existence of equilibria for this market. Equilibrium prices naturally emerge in the norm dual of $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$, i.e., the set of bounded additive set functions absolutely continuous w.r.t. μ. But, a price system which is only finitely additive has no economic interpretation: it makes an arbitrary small set of commodities extraordinarily expensive. Thus, special care must be taken to ensure that the equilibrium price system is countably additive.

To apply Bewley's result, some topological requirements must be satisfied by each consumption set $X_{(i,E_i)}$, the production possibility set Y, and each consumer's preference mapping from $X_{(i,E_i)}$ to R. The *weak-star topology* on $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ is defined to be the topology of pointwise convergence on $\mathscr{L}_1(\Omega, \mathscr{H}, \mu)$; that is, if $\boldsymbol{x}^\lambda$ is a *net* in $\mathscr{L}_\infty$, then $\boldsymbol{x}^\lambda \to \boldsymbol{x}$ iff

$$\int \boldsymbol{f}(w)\boldsymbol{x}^\lambda(w)d\mu(w) \to \int \boldsymbol{f}(w)\boldsymbol{x}(w)d\mu(w), \qquad \forall \boldsymbol{f} \in \mathscr{L}_1.[17]$$

The *Mackey topology* on $\mathscr{L}_\infty(\Omega, \mathscr{H}, \mu)$ is defined to be the topology of uniform convergence on weak-star compact, convex circled subsets of $\mathscr{L}_1(\Omega, \mathscr{H}, \mu)$.[18] Clearly, the Mackey topology is stronger than the weak-star topology; that is, if $\boldsymbol{x}^\lambda \to \boldsymbol{x}$ in the Mackey topology, then $\boldsymbol{x}^\lambda \to \boldsymbol{x}$ also in the weak-star topology. In other words, any weak-star closed subset of $\mathscr{L}_\infty$ is Mackey closed. Bewley requires that each $X_{(i,E_i)}$ and Y are Mackey closed and the preference mapping of each consumer is Mackey continuous.

It is easy to see that $X_{(i,E_i)}$ is weak-star closed; thus, it is also Mackey closed. The following two lemmas prove the two remaining requirements.

Lemma 1. *Under Assumptions* 2 *and* 3*, Y is Mackey closed.*

Proof: Let $\boldsymbol{y}^\lambda$ be a net in Y such that $\boldsymbol{y}^\lambda \to \boldsymbol{y}$ in the weak-star topology. By the definition of Y, for each $\boldsymbol{y}^\lambda$ there exists $\boldsymbol{\delta}_\lambda^N \in \Delta(N)$ such that $\boldsymbol{y}^\lambda(w) \le \boldsymbol{z}^N(\boldsymbol{\delta}_\lambda^N(w), w)$ a.e. Since $\Delta(N)$ is compact, we can take a subnet $\boldsymbol{\delta}_{\lambda'}^N$ which converges to some $\boldsymbol{\delta}^N \in \Delta(N)$ in the norm topology. We then get $\boldsymbol{z}^N[\boldsymbol{\delta}_{\lambda'}^N] \to \boldsymbol{z}^N[\boldsymbol{\delta}^N]$ in the weak-star topology, since $\boldsymbol{\delta}_{\lambda'}^N \to \boldsymbol{\delta}^N$ in the norm topology implies that $\boldsymbol{z}^N(\boldsymbol{\delta}_{\lambda'}^N(w), w) \to \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)$ a.e., and so $\int \boldsymbol{z}^N(\boldsymbol{\delta}_{\lambda'}^N(w), w)\boldsymbol{f}(w)d\mu(w) \to \int \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)\boldsymbol{f}(w)d\mu(w), \forall \boldsymbol{f} \in \mathscr{L}_1$. Since the nonnegative orthant of $\mathscr{L}_\infty$ is weak-star closed, it follows that $\boldsymbol{y}(w) \le \boldsymbol{z}^N(\boldsymbol{\delta}^N(w), w)$ a.e. Therefore, $\boldsymbol{y} \in Y$. From this we conclude that Y is weak-star closed, and hence Mackey closed. ■

[17] For the definition of a net, or a *generalized sequence*, see Dunford and Schwarz [8, p. 26].

[18] See Bewley [5] for a more precise definition of the Mackey topology.

We must add the following assumption to ensure the Mackey continuity of the consumers' preferences.

Assumption 5. *For each i and for each scalar x, there exists a function $\boldsymbol{f}$ in $\mathscr{L}_1(\Omega, \mathscr{H}, \mu)$ for which $\boldsymbol{u}_i(x, w) \leq \boldsymbol{f}(w)$ a.e.*

Lemma 2. *Under Assumptions 1, 3, and 5, the preferences mapping of each consumer (i, E_i), which is a mapping from $X_{(i,E_i)}$ to R defined by*

$$\int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w) d\mu_i(w)/\mu_i(E_i)\,, \quad \forall \boldsymbol{x} \in X_{(i,E_i)}\,,$$

is Mackey continuous.

Proof: It is sufficient to prove that, for each (i, E_i), if a net $\boldsymbol{x}^\lambda$ in $X_{(i,E_i)}$ converges to $\boldsymbol{x} \in X_{(i,E_i)}$ in the weak-star topology and either $\boldsymbol{x}^\lambda(w) \leq x(w)$ a.e., $\forall\lambda$ or $\boldsymbol{x}^\lambda(w) \geq \boldsymbol{x}(w)$ a. e., $\forall\lambda$, then

$$\int_{E_i} |\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w)| d\mu_i(w) \to 0.$$[19]

If $\boldsymbol{x}^\lambda \to \boldsymbol{x}$ in the weak-star topology and $\boldsymbol{x}^\lambda(w) \leq \boldsymbol{x}(w)$ a.e., $\forall\lambda$, then for any $\boldsymbol{x}' \in \mathscr{L}_\infty^{b_i}(\Omega, \mathscr{H}, \mu)$ we have

$$\begin{aligned}
0 &\leq \int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}'(w), w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) \\
&\leq \int_{E_i} \boldsymbol{u}_i(\|\boldsymbol{x}'\|_\infty, w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) \\
&\leq \int_{E_i} \boldsymbol{f}(w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) \\
&\leq n \int_{E_i} \boldsymbol{f}(w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu(w).
\end{aligned}$$

Since $\boldsymbol{f} \in \mathscr{L}_1(\Omega, \mathscr{H}, \mu)$, the last integral converges to zero by the definition of convergence in the weak-star topology; thus,

$$\int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}'(w), w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) \to 0.$$

The same result holds if $\boldsymbol{x}^\lambda \to \boldsymbol{x}$ and $\boldsymbol{x}^\lambda(w) \geq \boldsymbol{x}(w)$ a.e., $\forall\lambda$.

Assume that $\boldsymbol{x}^\lambda(w) \leq \boldsymbol{x}(w)$ a.e., $\forall\lambda$. If $\boldsymbol{x}(w) > b_i$, the concavity and monotonicity of $\boldsymbol{u}_i(x, w)$ in x implies

$$\boldsymbol{u}_i(\boldsymbol{x}(w), w) - \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) \leq \frac{\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)}{\boldsymbol{x}(w) - b_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w).$$

[19] See Bewley [5, pp. 535-536] for this argument.

If $\boldsymbol{x}(w) = b_i$, then $\boldsymbol{x}^\lambda(w) = b_i$ and so $\boldsymbol{u}_i(\boldsymbol{x}(w), w) - \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) = 0$. For each $t > 0$, let $B_t = \{w \in E_i : b_i < \boldsymbol{x}_i(w) < b_i + t\}$ and $C_t = \{w \in E_i : \boldsymbol{x}_i(w) \geq b_i + t\}$. Then, using the above inequality, we get

$$\begin{aligned}
&\int_{E_i} (\boldsymbol{u}_i(\boldsymbol{x}(w), w) - \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w)) d\mu_i(w) \\
&\leq \int_{B_t} (\boldsymbol{u}_i(\boldsymbol{x}(w), w) - \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w)) d\mu_i(w) \\
&\qquad\qquad + \int_{C_t} \frac{1}{t} \boldsymbol{u}_i(\boldsymbol{x}(w), w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) \\
&\leq \int_{B_t} (\boldsymbol{u}_i(\boldsymbol{x}(w), w) d\mu_i(w) + \frac{1}{t} \int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w).
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrarily given. Since $\chi_{B_t}(w) \to 0, \forall w \in E_i$, as $t \to 0$, there exists t such that $\int_{B_t} \boldsymbol{u}_i(\boldsymbol{x}(w), w) d\mu_i(w) < \varepsilon/2$ by the dominated convergence theorem. Further, we can choose λ_1 so large that if $\lambda \geq \lambda_1$ then $\int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w)(\boldsymbol{x}(w) - \boldsymbol{x}^\lambda(w)) d\mu_i(w) < \varepsilon t/2$.Then, if $\lambda \geq \lambda_1$,

$$\int_{E_i} (\boldsymbol{u}_i(\boldsymbol{x}(w), w) - \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w)) d\mu_i(w) < \varepsilon.$$

Assume that $\boldsymbol{x}^\lambda(w) \geq \boldsymbol{x}(w)$ a.e.,$\forall \lambda$. Let $t > 0$. If $\boldsymbol{x}(w) \geq b_i + t$, then

$$\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w) \leq \frac{\boldsymbol{x}^\lambda(w) - \boldsymbol{x}(w)}{t} \boldsymbol{u}_i(\boldsymbol{x}(w), w).$$

If $\boldsymbol{x}(w) < b_i + t < \boldsymbol{x}^\lambda(w)$, then

$$\begin{aligned}
&\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w) \\
&= \boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(b_i + t, w) + \boldsymbol{u}_i(b_i + t, w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w) \\
&\leq \frac{\boldsymbol{x}^\lambda(w) - b_i - t}{t} \boldsymbol{u}_i(b_i + t, w) + \boldsymbol{u}_i(b_i + t, w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w) \\
&\leq \frac{\boldsymbol{x}^\lambda(w) - x(w)}{t} \boldsymbol{u}_i(b_i + t, w) + \boldsymbol{u}_i(b_i + t, w)
\end{aligned}$$

If $x^\lambda(w) \leq b_i + t$, then

$$\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w) \leq \boldsymbol{u}_i(b_i + t, w)$$

So we get

$$\begin{aligned}
&\int_{E_i} (\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w)) d\mu_i(w) \\
&\leq \frac{1}{t} \int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w)(\boldsymbol{x}^\lambda(w) - \boldsymbol{x}(w)) d\mu_i(w) \\
&+ \frac{1}{t} \int_{E_i} \boldsymbol{u}_i(b_i + t, w)(\boldsymbol{x}^\lambda(w) - \boldsymbol{x}(w)) d\mu_i(w) \\
&+ \int_{E_i} \boldsymbol{u}_i(b_i + t, w) d\mu_i(w).
\end{aligned}$$

Since $\boldsymbol{u}_i(b_i, w) = 0$, $\forall w$, and $\boldsymbol{u}_i(x, w)$ is continuous and nondecreasing in x, $\int_{E_i} \boldsymbol{u}_i(b_i + t, w) d\mu_i(w) \to 0$ as $t \to 0$ by the dominated convergence theorem. Thus, for any given $\varepsilon > 0$, take t so small that $\int_{E_i} \boldsymbol{u}_i(b_i + t, w) d\mu_i(w) < \varepsilon/3$. Choose λ_1 such that if $\lambda \geq \lambda_1$,

$$\int_{E_i} \boldsymbol{u}_i(\boldsymbol{x}(w), w)(\boldsymbol{x}^\lambda(w) - \boldsymbol{x}(w)) d\mu_i(w) < t\varepsilon/3$$

and

$$\int_{E_i} \boldsymbol{u}_i(b_i + t, w)(\boldsymbol{x}^\lambda(w) - \boldsymbol{x}(w)) d\mu_i(w) < t\varepsilon/3$$

Then, if $\lambda \geq \lambda_1$,

$$\int_{E_i} (\boldsymbol{u}_i(\boldsymbol{x}^\lambda(w), w) - \boldsymbol{u}_i(\boldsymbol{x}(w), w)) d\mu_i(w) < \varepsilon.$$

■

Our final assumption replaces Bewley's Adequacy assumption.

Assumption 6. *For each i, there exists a strategy $\boldsymbol{\delta}$ in $\Delta(\{i\})$ such that $\boldsymbol{z}^{\{i\}}(\boldsymbol{\delta}(w), w) > b_i + \varepsilon_i$ a.e. for some $\varepsilon_i > 0$.*

Theorem 3. *Let $\boldsymbol{d} = \{\boldsymbol{d}_i\}_{i \in N}$ be a core-compatible budget plan, satisfying for each i and w:* (a) *$\boldsymbol{d}_i(\boldsymbol{\pi}, w)$ is a continuous function of $\boldsymbol{\pi}$ on Π w.r.t. the norm topology of $ba(\Omega, \mathscr{H}, \mu)$;* (b) *if $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ in Π satisfy the inequality*

$$\sup_{\boldsymbol{\delta} \in \Delta(N)} \int \boldsymbol{z}^N(\boldsymbol{\delta}(w), w) d\boldsymbol{\pi}(w) \geq \sup_{\boldsymbol{\delta} \in \Delta(N)} \int \boldsymbol{z}^N(\boldsymbol{\delta}(w), w) d\boldsymbol{\pi}'(w),$$

then $\boldsymbol{d}_i(\boldsymbol{\pi}, w) \geq \boldsymbol{d}_i(\boldsymbol{\pi}', w)$,and (c) *if $\{\boldsymbol{\pi}_\lambda\}$ is a net in Π such that*

$$\sup_{\boldsymbol{\delta} \in \Delta(N)} \int \boldsymbol{z}^N(\boldsymbol{\delta}(w), w) d\boldsymbol{\pi}_\lambda(w) \to \sup_{\boldsymbol{\delta} \in \Delta(N)} \int \boldsymbol{z}^N(\boldsymbol{\delta}(w), w) d\boldsymbol{\pi}(w),$$

then $\boldsymbol{d}_i(\boldsymbol{\pi}_\lambda, w) \to \boldsymbol{d}_i(\boldsymbol{\pi}, w)$. Under Assumptions 1, 2, 3, 5, 6, *there exists a constrained competitive equilibrium contract relative to $\boldsymbol{d}$, in which $\bar{\boldsymbol{\pi}} \in \Pi$ satisfies $\bar{\boldsymbol{\pi}}(\mathscr{G}_i(w)) > 0$ for each i and w.*

Proof: This is a direct consequence of applying Bewley's Theorem 1 (which establishes existence of an equilibrium with a price system $\bar{\boldsymbol{\pi}}'$ in $ba(\Omega, \mathscr{H}, \mu)$) and Theorem 3 (which ensures existence of a countably additive price system $\bar{\boldsymbol{\pi}}$ corresponding to $\bar{\boldsymbol{\pi}}'$). The monotonicity assumption is satisfied due to Assumptions 1 and 3, and the boundedness assumption is satisfied due to condition (iv) of Assumption 2. Bewley's proof of Theorem 1 must be modified to adjust for the fact that the budget plan $\boldsymbol{d}$ may not take the form that each consumer gets a constant proportion of the producer's profit. This modification is straightforward if $\boldsymbol{d}$ satisfies (a), (b), and (c). Assumption 6 and the core-compatibility of $\boldsymbol{d}$ implies

$$\boldsymbol{d}_i(\boldsymbol{\pi}, w) > \inf_{\boldsymbol{x} \in X_{(i, \mathscr{G}_i(w))}} \mathscr{E}_{\boldsymbol{\pi}} \{\boldsymbol{x} | \mathscr{G}_i\}(w)$$

for each i and w if $\boldsymbol{\pi}(\mathscr{G}_i(w))$ is positive; hence, the implied allocation of each consumer $(i, \mathscr{G}_i(w))$ attains maximal preference in his budget set. Thus, there exists an equilibrium contract with $\bar{\boldsymbol{\pi}}'$ in $ba(\Omega, \mathscr{H}, \mu)$. Due to Yoshida-Hewitt Theorem A (Bewley [5, p. 534]), $\bar{\boldsymbol{\pi}}'$ can be decomposed uniquely into a countably additive set function $\bar{\boldsymbol{\pi}}$ and a purely finitely additive set function $\bar{\boldsymbol{\pi}}_p$ in such a way that $\bar{\boldsymbol{\pi}}' = \bar{\boldsymbol{\pi}} + \bar{\boldsymbol{\pi}}_p$. Bewley's Exclusion Assumption is satisfied due to Yoshida-Hewitt Theorem B (Bewley [5, p. 534]), so that his Theorem 3 implies that this $\bar{\boldsymbol{\pi}}$ serves as an equilibrium price system for the previously implied contract.[20]

■

8 An Example

A syndicate is formed by n agents for an opportunity to invest their capital in a risky project. The syndicate can finance additional funds by borrowing or can invest a portion of their capital in a riskless asset in any amount at a fixed risk-free rate of interest. The problem is to determine the size of the project and the rule to split the syndicate's final wealth among the members.

First, we develop the problem in accordance with the formulation of Section 2. The basic uncertainty faced by the agents is associated with the rate of return of the project. Let $\boldsymbol{\theta}$ be the project's rate of return, taking values in a subset Θ of the real line. We assume that the size of the project does not affect the stochastic nature of $\boldsymbol{\theta}$. An agent i contributes w_i dollars to the formation of the syndicate's capital. If α is the proportion of capital invested in the project and r is the (borrowing and lending) rate of interest, the syndicate's final wealth is given as $\boldsymbol{z}^N(\alpha, \theta) = \{(1+r)+\alpha(\theta-r)\} \sum_{i\in N} w_i$. The value of α is an arbitrary nonnegative number, where a value of α larger than one means that they invest more than their capital in the project and a value of α smaller than one means that they invest a portion of their capital in the riskless asset.

Differential information is introduced in the following way. Before the initial date each agent has observed a private sample which is correlated with $\boldsymbol{\theta}$. The sample observed by an agent i is denoted by $\boldsymbol{s}_i$, whose value is in a set S_i. The state space Ω is then taken to be the product space $\Theta \times \times_{i\in N} S_i \equiv \Theta \times S$, whose elements are of the form $(\boldsymbol{\theta}, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n) = (\boldsymbol{\theta}, \boldsymbol{s})$. We assume for simplicity that the agents obtain no further information before the date of action, at which time the values of the samples are revealed to each other for the purpose of investment. The project's realized rate of return is known to everyone at the final date.

Thus, the informational field $\mathscr{H}$ which is common to everyone at the date of contract delivery is the σ-field generated by $(\boldsymbol{\theta}, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$. A strategy $\boldsymbol{\delta}^N$ of the syndicate is a function from S to R, which reflects the measurability condition

[20] In his Theorem 3 Bewley requires that the production possibility set be a cone, but one can easily strengthen his proof in such a way that this assumption becomes unnecessary.

at the date of action. A sharing plan $\boldsymbol{x}^N$ is an n-tuple of functions $\boldsymbol{x}_i^N$ from $\Theta \times S$ to R. A contract $\langle \boldsymbol{\delta}^N, \boldsymbol{x}^N \rangle$ is feasible for the syndicate iff it satisfies the inequality $\sum_{i \in N} \boldsymbol{x}_i^N(\theta, s) \leq \boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta)$ for each $\theta \in \Theta$ and $s \in S$.

We assume that the distribution of $(\boldsymbol{\theta}, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$, which is assessed by each agent, has a probability density (or a probability mass function, if these are discrete random variables). Further, the density of $(\boldsymbol{\theta}, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ of agent i is expressed as the product of $\phi_i(\theta)$ and $l(s_1, \ldots, s_n|\theta)$. Namely, the agents may have differing prior densities of $\boldsymbol{\theta}$, but they agree on the assessment of the conditional density of $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ given $\boldsymbol{\theta}$.

To determine a core-compatible budget plan, we must make an assumption on the investment opportunity available to each subcoalition. The simplest one is to assume that each subcoalition has no opportunity other than to invest its capital in the riskless asset. The budget of agent i is a function $\boldsymbol{d}_i$ from $\Pi \times S_i$ to R, and Definition 7 yields

$$\boldsymbol{d}_i(\boldsymbol{\pi}, s_i) \leq (1+r)w_i, \quad \forall i \in N.$$

On the other hand, condition (ii) of Definition 6 implies that an equilibrium $\boldsymbol{\pi}$ must satisfy $\mathscr{E}_{\boldsymbol{\pi}}\{\boldsymbol{\theta} - r\} = 0$; i.e., the maximum expected wealth achievable by the syndicate equals $(1+r)\sum_{i \in N} w_i$. From Definition 5 it then follows that a core-compatible budget plan is given as

$$\boldsymbol{d}_i(\boldsymbol{\pi}, s_i) = (1+r)w_i, \quad \forall i \in N.$$

Thus, in this example we can determine uniquely a core-compatible budget plan. This is due to the assumption that the stochastic nature of the project's rate of return is independent of the size of the project and the syndicate can borrow or lend capital at a fixed interest rate without limit; that is, the syndicate's investment opportunity has the property of *stochastic constant returns to scale.*

The budget plan derived above is actually valid for a wider class of assumptions on the investment opportunity of each subcoalition. Assume that each subcoalition at least has the riskless opportunity with the interest rate r. Then, a core-compatible budget plan, if it exists, must have this form due to Definitions 5 and 7. If, for example, the risky project that the syndicate has is also available to each subcoalition, then this budget plan satisfies all the required inequalities of Definition 7. A more general condition was given by Theorem 1.

Logarithmic Utility Functions

Assume that each agent i has the state-independent utility function $u_i(x) = \log(x - b_i)$, in which b_i is a constant to be interpreted as his minimum wealth for subsistence.[21] The risk *aversion function*[22] of u_i is given as

[21] If an agent i has additional personal income, it should be incorporated in the coefficient b_i. We assume that no agent has additional personal income which is subject to risk.

[22] See Pratt [14] and Arrow [2].

$-u_i''(x)/u_i'(x) = (x - b_i)^{-1}$, the reciprocal of a linear function of x. The coefficient of x in this linear function is called the *risk cautiousness*; the risk cautiousness of an agent having a logarithmic utility function is one. As for private information, we first consider the restrictive situation described below. The set of agents N is the union of disjoint sets J and K. Each agent in J observes the outcome of a common sample $\boldsymbol{s}$. The agents in K do not observe $\boldsymbol{s}$ nor any other information. That is, the set J consists of informed agents, while the set K consists of uninformed agents. This simplifying assumption is made to study the implication of the presence of differential information for the equilibrium contracts in its purest form.

The equilibrium, derived as the solution to (i), (ii), (iii) of Definition 6, is the following. Let g_i be agent i's unconditional probability density of $\boldsymbol{s}$ (i.e., $g_i(s) = \int_\Theta \phi_i(\theta) l(s|\theta) d\theta$), and let $f_i(\theta|s)$ be his posterior density of $\boldsymbol{\theta}$ given $\boldsymbol{s}$ (i.e.,$f_i(\theta|s) = \phi_i(\theta) l(s|\theta)/g_i(s)$). For simplicity, denote that $w_J = \sum_{j\in J} w_j, w_K = \sum_{k\in K} w_k, w_0 = w_J + w_K$, and similarly that $b_J = \sum_{j\in J} b_j, b_K = \sum_{k\in K} b_k, b_0 = b_J + b_K$. Let $g_K(s)$ be the weighted average of the unconditional densities of $\boldsymbol{s}$ of the uninformed members, defined by

$$g_K(s) = \sum_{k\in K} [(1+r)w_k - b_k] g_k(s) / [(1+r)w_K - b_K].$$

Define $f_J(\theta|s)$ and $f_K(\theta|s)$, respectively, by

$$f_J(\theta|s) = \frac{\sum_{j\in J}[(1+r)w_j - b_j] f_j(\theta|s)}{(1+r)w_J - b_J}$$

and

$$f_K(\theta|s) = \frac{\sum_{k\in K}[(1+r)w_k - b_k] g_k(s) f_k(\theta|s)}{[(1+r)w_K - b_K] g_K(s)}$$

and let $f_0(\theta|s)$ be

$$f_0(\theta|s) = \frac{[(1+r)w_J - b_J] f_J(\theta|s) + [(1+r)w_K - b_K] f_K(\theta|s)}{(1+r)w_0 - b_0}.$$

Then, the equilibrium strategy is the $\boldsymbol{\delta}^N$ that satisfies

$$\int_\Theta \frac{\theta - r}{\boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) - b_0} f_0(\theta|s) d\theta = 0,$$

and the equilibrium sharing plan is given as

$$\begin{aligned} \boldsymbol{x}_j^N(\theta, s) &= b_j + \frac{[(1+r)w_j - b_j] f_j(\theta|s)}{[(1+r)w_0 - b_0] f_0(\theta|s)} [\boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) - b_0], \qquad \forall j \in J; \\ \boldsymbol{x}_k^N(\theta, s) &= b_k + \frac{[(1+r)w_k - b_k][g_k(s)/g_K(s)] f_k(\theta|s)}{[(1+r)w_0 - b_0] f_0(\theta|s)} \\ &\quad \times [\boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) - b_0], \qquad \forall k \in K. \end{aligned}$$[23]

The functions $f_J(\theta|s)$, $f_K(\theta|s)$, and $f_0(\theta|s)$ are interpreted as the *surrogate* posterior densities of $\boldsymbol{\theta}$ given $\boldsymbol{s}$ of the informed, of the uninformed, and of all the members, respectively. The syndicate's posterior density is constructed by taking an arithmetic weighted average of individual posterior densities. The probability judgment of each member is reflected in $f_0(\theta|s)$ in proportion to his *net capital contribution*, which is defined for member i as $(1+r)w_i - b_i$. The weight of an uninformed member k is further proportional to $g_k(s)/g_K(s)$: the probability he assigns to the event $\{\boldsymbol{s} = s\}$ relative to the average probability among the uninformed members. Thus, the opinion of an uninformed member gets a higher weight in an event to which he assigns a higher probability prior to communication.

The condition specifying the equilibrium strategy can be rewritten as

$$\frac{d}{d\alpha}\int_{\Theta} \log[\boldsymbol{z}^N(\alpha,\theta) - b_0] f_0(\theta|s) d\theta\big|_{\alpha=\boldsymbol{\delta}^N(s)} = 0.$$

Namely, the syndicate's equilibrium strategy is to choose, for each outcome of the sample $\boldsymbol{s}$, the size of the project that attains the maximum expected utility of the syndicate's wealth with respect to the density $f_0(\theta|s)$.[24] The *surrogate* utility function to be used is the logarithmic function whose subsistence coefficient equals the sum b_0 of individual subsistence coefficients.[25]

The equilibrium sharing plan is to split the syndicate's wealth in excess of the aggregate subsistence level among the members in a linear fashion, in addition to paying each a sure income of the amount equal to his subsistence wealth. The rule involves *bets* due to differences in the members' subjective probability judgments. Each member obtains, in a state (θ, s), a share of wealth in proportion to his net capital contribution and his probability assessment of the event $\{\boldsymbol{\theta} = \theta\}$ given $\{\boldsymbol{s} = s\}$. The share of an uninformed member k is further proportional to $g_k(s)/g_K(s)$.

If there is agreement on the prior probability assessment of $\boldsymbol{\theta}$ among the *uninformed* members, then the equilibrium reduces to the following. The syndicate's posterior density of $\boldsymbol{\theta}$ is

$$f_0(\theta|s) = \frac{\sum_{i\in N}[(1+r)w_i - b_i] f_i(\theta|s)}{(1+r)w_0 - b_0},$$

[23] The density of the equilibrium $\boldsymbol{\pi}$ is given as

$$[(1+r)w_0 - b_0] g_K(s) \frac{f_0(\theta|s)}{z^N(\boldsymbol{\delta}^N(s),\theta) - b_0}.$$

[24] If the expected utility is monotone increasing in α, then there exists no equilibrium. The situation dealt with in the example is not covered by the existence theorem (Theorem 4), since the action space is not compact.

[25] The decomposition of "group preferences" into surrogate utility and surrogate probability functions, as observed here, is of a more general nature. See Wilson [20] and Kobayashi [10].

and the wealth allocated to agent i is

$$\boldsymbol{x}_i^N(\theta, s) = b_i + \frac{[(1+r)w_i - b_i]f_i(\theta|s)}{[(1+r)w_0 - b_0]f_0(\theta|s)}[\boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) - b_0], \qquad \forall i \in N.$$

One can easily verify that this coincides with the equilibrium contract for the situation in which the informed members reveal the value of the sample to the uninformed members at or before the initial date (*full-communication equilibrium*). This result is not surprising. The basic feature of the notions of the conditional core and that of the constrained competitive equilibrium is to allow informationally inferior agents to insure collectively against the proprietary information held by others. It is the possibility of such mutual insurance activities that distinguishes the constrained competitive equilibrium from the full-communication equilibrium. If all the uninformed members agree on the probability judgment of the sample $\boldsymbol{s}$, there remains no room for them to bet on $\boldsymbol{s}$. In other words, agreement on the probability assessment of $\boldsymbol{s}$ among the uninformed members excludes the opportunity to insure among themselves against the commonly unknown sample.[26]

As one can predict, the previous conclusion can be extended to the more general situation in which the members observe different private samples. That is, if all the members have identical prior probability judgment of $\boldsymbol{\theta}$, then the equilibrium contract coincides with the full-communication equilibrium contract. The strategy is the one that maximizes, for each outcome of the joint sample, the conditional expected utility of the syndicate's wealth. It is to be calculated by using the logarithmic utility function with the subsistence coefficient b_0 and the common posterior distribution of $\boldsymbol{\theta}$ given the joint sample. The sharing plan is to allocate each member a sure income which equals his subsistence wealth and to split the syndicate's wealth in excess of b_0 in proportion to the members' net capital contributions.

Exponential Utility Functions

Consider the case in which all the agents have exponential utility functions. The utility function of agent i is $u_i(x) = -\exp(-x/\rho_i)$, where ρ_i is a positive number to be called the *risk tolerance* coefficient.[27] First, we exhibit the nature of the equilibrium strategy for the general case in which the agents observe different private samples. Define $f_0(\theta|s)$ by

$$f_0(\theta|s) = c \prod_{i \in N} [f_i(\theta|s)]^{\rho_i/\rho_0},$$

[26] If the members had additional personal wealths that were random and correlated with the sample, then there would still be opportunities for the uninformed members to share the private risks across different values of the random sample. The equilibrium, then, would be different from the full-communication equilibrium.

[27] This utility function satisfies the property $-u_i''(x)/u_i'(x) = \rho_i^{-1}$; namely, the function has constant risk aversion. The risk cautiousness of u_i is zero.

where $\rho_0 \equiv \sum_{i \in N} \rho_i$ and c is a normalization constant. Then, the equilibrium strategy is given by

$$\boldsymbol{\delta}^N(s) \in \arg\max_{\alpha \geq 0} \int_{\Theta} -\exp[-\boldsymbol{z}^N(\alpha, \theta)/\rho_0] f_0(\theta|s) d\theta$$

for each outcome of the joint sample $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$.[28] After revealing the values of the private samples, the syndicate chooses the size of the project so as to maximize the conditional expected utility of its final wealth. The syndicate's posterior density of $\boldsymbol{\theta}$ is constructed by taking a geometric average of individual posterior densities, in which each member is given a weight proportional to the magnitude of his risk tolerance. The syndicate's utility function is the exponential function whose risk tolerance equals the sum of all the individual risk tolerances.

The equilibrium sharing plan is of the form

$$\boldsymbol{x}_i^N(\theta, s) = \frac{\rho_i}{\rho_0} \boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) + \rho_i \log\left[\frac{\phi_i(\theta)}{\phi_0(\theta)}\right] + \rho_i \log\left[\frac{\lambda_i(s_i)}{\prod_{j \in N}[\lambda_j(s_j)]^{\rho_i/\rho_0}}\right],$$

where $\phi_0(\theta) \equiv \Pi_{i \in N}(\phi_i(\theta))^{\rho_i/\rho_0}$ and λ_i is a known function of s_i.[29] The share of each agent is the sum of a*dividend* and two additional random payments. The dividend is determined by the condition that the agents split the syndicate's wealth in proportion to the magnitudes of their risk tolerances. The middle term reflects a *side bet* on $\boldsymbol{\theta}$. The last term is a random payment which depends on the realized values of the private samples.

The peculiar features of the equilibrium contract, in comparison with the equilibrium contract implied by logarithmic utilities, are two-fold. The weight of each agent's probability assessment in constructing the syndicate's assessment is solely determined by his attitude toward risk, and is not affected by the amount of capital he contributes to the syndicate. Further, the aggregate risk tolerance is the sum of the individual risk tolerances, so that the syndicate can bear more risk as the number of members increases. These properties depend critically on the result that when the members have constant risk tolerances, efficient sharing of risk requires that the risk be shared among the individuals in proportion to their risk tolerances.[30]

It is worth noting that the equilibrium strategy coincides with the one associated with the full-communication equilibrium. That is, in the case of exponential utilities, the presence of differential information does not affect the form of the equilibrium strategy; it does not make any difference whether the agents reveal

[28] See footnote 25.

[29] It is important that λ_i is a function of s_i but independent of the other samples. It captures the peculiar requirement of efficiency with differential information that the marginal utility of income of each agent be measurable with respect to his information (see Kobayashi [10]).

[30] For efficiency risk should be shared at the margin in proportion to risk tolerances, no matter what the utility functions are. This basic property of efficient risk sharing was first discovered by Wilson [20] (see also Wilson [21]).

their private information before or after the contract is established.[31] If there is agreement on the prior probability assessment of $\boldsymbol{\theta}$ among all the members, then the equilibrium sharing plan also coincides with the one associated with the full-communication equilibrium, and it is given by

$$\boldsymbol{x}_i^N(\theta, s) = \frac{\rho_i}{\rho_0}\boldsymbol{z}^N(\boldsymbol{\delta}^N(s), \theta) + (1+r)\left[w_i - \frac{\rho_i}{\rho_0}w_0\right].$$

9 Conclusion

The objective of this paper was to develop a conceptual framework for solving the syndicate problem when agents have different information about the state of nature. The basic solution concept was the conditional core. The constrained competitive equilibrium contracts were shown to be core-compatible with the choice of an appropriate budget plan. The analysis of an example showed that the equilibrium contracts have appealing properties when every individual agent has unit risk cautiousness and when every individual agent has zero risk cautiousness. It was conjectured that the conditional core shrinks to the set of equilibrium contracts as the number of agents participating in the syndicate approaches to infinity.

The definition of the conditional core was based on the assumption that the agents are reluctant to reveal their private information when they negotiate on the terms of a contract. As such, the conditional core is the one that retains the greatest opportunity for mutual insurance subject to the limitation inherent in the presence of differential information. An explicit consideration of the strategic aspects of information, such as problems of moral hazard and adverse selection, would require formulation as a noncooperative game.

The concept of the constrained competitive equilibria may be regarded as a substitute for the Arrow-Debreu equilibria for economies under uncertainty in which traders come to the market with different prior information about the environment. The critical idea was to prohibit inside trades: a trader is not allowed to sell a contract for delivery contingent on an event that he already knows could not occur. A later paper will study the institutional form of a market mechanism which achieves a core allocation of risk with the presence of differential information, in relation to differing degrees of completeness of the market.

[31]This was also the case when the utility functions were logarithmic and there was agreement on the prior probability assessment of $\boldsymbol{\theta}$. The general condition for such coincidence is given in Kobayashi [10].

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