COURNOT COMPETITION IN A DIFFERENTIATED OLIGOPOLY

K. Sridhar Moorthy*

Abstract

This paper studies product-quantity equilibria in an oligopoly. Products are interpreted as "qualities" and each firm chooses a quality-quantity pair, simultaneously. It is well known that a pure-strategy equilibrium in product-price pairs does not exist in this model, but a pure-strategy equilibrium in product-quantity pairs exists. Furthermore, in an example widely studied in the literature, the equilibrium has nice asymptotic properties.

1 Introduction

In this paper I study product-quantity equilibria in an oligopoly. Products are interpreted as qualities (*i.e.,* there exists an ordering of products), and each firm chooses a product and its supply quantity, simultaneously. I focus primarily on the necessary conditions for a pure-strategy Nash equilibrium, although in an example I verify that these conditions are sufficient as well. The necessary conditions have a recursive structure, and can be solved for the "local" equilibrium—an equilibrium where the ordering of the firms' products is exogenously fixed. Once the local equilibrium is computed, verifying that it is a Nash equilibrium amounts to checking that no firm wants to change its position in the product ordering unilaterally. It is the first part of this procedure that complicates matters in the Bertrand case. Because no pure-strategy Nash equilibrium exists when firms choose products and prices simultaneously, we

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modify the strategy space to allow firms to choose products before prices. But then the necessary conditions for the first-stage local product equilibrium form a non-recursive set of equations, difficult to solve; cf. Moorthy [15].

I find that the Cournot equilibrium is asymmetric in general—different firms choose different products and supply different quantities—even though the firms are identical. It is also essentially unique, and inefficient. As the number of firms approaches infinity, however, the equilibrium becomes efficient; moreover, in the example, the distribution of products converges weakly to the distribution of consumer types (assumed to be uniform in this paper).

Previous work on differentiated oligopolies has mostly been in the Bertrand framework, in Hotelling-type [13] location models; cf. Prescott and Visscher [18], d'Aspremont, Gabszewicz, and Thisse [5], and Novshek [17]. The distinguishing feature of such models is that every type of consumer has a different favorite product even when all products are priced the same. In contrast, my model is a "quality-choice" model: products are differentiated on a single dimension which behaves like "quality"—each consumer prefers higher values on this dimension to lower values—and consumers differ in their marginal willingness to pay for quality.

Gabszewicz and Thisse [7] have looked at the price equilibrium with qualities exogenously fixed. Gabszewicz and Thisse [8], in addition, study the asymptotic behavior of the price equilibrium as the number of firms increases. But their entry process is rather special: each new firm enters at a prespecified fixed distance from the highest existing quality. They find that the price equilibrium tends to the perfectly competitive outcome with entry even when there is an upper bound on the number of firms that can have positive equilibrium market shares. In my model, by contrast, product locations are determined as part of an equilibrium, and these locations are not equispaced, except in the limit. I do not have an upper bound on the number of firms that can have positive market shares either, because each type of consumer has a distinct favorite product under marginal-cost pricing; cf. Shaked and Sutton [20]. Shaked and Sutton [19] have shown the existence of a pure-strategy equilibrium in a duopoly model, where firms choose products before prices and marginal costs of the firms are independent of quality. Moorthy [15] analyzes the corresponding case with marginal costs increasing in quality. Bresnahan [3] allows firms to choose multiple products before prices, and proves the existence of a mixed-strategy equilibrium.

In the Cournot framework, the only related works I am aware of are Gal-Or [9] and Hart [10,11]. Gal-Or's model of the consumer is similar to mine, except she does not model substitute product classes explicitly, as I do. But there is a crucial difference between our papers. Gal-Or allows each firm to choose a continuum of products, whereas in my model, each firm chooses a single product.

Hart identifies conditions for the asymptotic efficiency of Cournot general-equilibria in a differentiated products economy. Although my model differs from his—my model is a partial-equilibrium model, has indivisible products, inelastic demand, and non-convex production sets (because of the one product per firm restriction)—it is instructive nevertheless to compare the reasons why our equilibria become efficient asymptotically. The key difference is the nature of the convergence. Hart replicates consumers of each type, thereby increasing each type's aggregate endowment, and enabling an increasing number of firms to exist in equilibrium. The equilibrium converges to efficiency because no firm, in the limit, supplies a substantial part of any consumer's consumption. I hold the consumer population fixed and replicate the firms directly. The reason for the asymptotic efficiency is simply the increasing substitutability among the firms' products..

The rest of the paper is organized as follows. In the next section I introduce the model. In Section 3 I define a Cournot–Nash equilibrium in products and quantities and note some of its basic properties. In the next two sections, I examine in turn two cases of my model: In Section 4 there is only one (low quality) substitute; in Section 5 there is a high quality substitute as well. In each of these cases I compute an example and study its properties. Section 6 concludes the paper.

2 The Model

In the most general version of the model—called the two-substitutes case hereafter—there are three product classes; two are called substitutes (lower and upper), and the third is the product class from which the n firms in the oligopoly choose products. (In the one-substitute case, there is only the lower substitute.) The substitutes delimit the feasible range of qualities for the firms: the lower substitute is the (greatest) lower bound on qualities, and the upper substitute, when it exists, is the (lowest) upper bound on qualities. But the essential difference between them and the firms is that the substitutes are available at fixed qualities and prices whereas the firms choose qualities and prices.[1] I will take the differentiated product class as $[s_0, \infty)$ in the one-substitute case (with $s_0 \geq 0$ as the lower substitute) and as $[s_0, s_{n+1}]$ in the two-substitutes case (with s_{n+1}as the upper substitute), with prices p_0 and p_{n+1} for the substitutes.

I assume a one-dimensional continuum of consumer types (segments) indexed by $t \in \Re$ and distributed uniformly on $[a, b], a < b$. I assume no income effects in the utility functions of these consumers so that their preferences can be parameterized by reservation prices. I further assume that each consumer buys just one unit of some product (*i.e.,* inelastic demand) or one of the substitutes. (These products are like durables.) Let $u(t, s)$ denote the reservation price of a type-t consumer for a unit of product $s \in \Re_+ = [0, \infty)$. The following properties of u are crucial to the construction[2]:

[1]The option of consuming the numéraire is always available to consumers in a partial equilibrium model such as ours. The substitutes, then, are other (more general) ways of closing the partial-equilibrium model. Their interpretation as "lower" and "upper" substitutes is natural given the interpretation of products as qualities.

[2]Hereafter, partial derivatives are denoted by subscripts (*e.g.,* $u_s(t, s) = \partial u(t, s)/\partial s$), and total derivatives by primes.

$$\forall t \in [a, b], \qquad \forall s \in [0, \infty), \qquad u_s(i, s) > 0,$$
$$u_t(t, s) > 0, \qquad u_{ts}(t, s) > 0. \tag{1}$$

Essentially, assumption (1) says that every consumer prefers a higher-quality product to a lower-quality product, but a higher type—a type with a larger t-index—attaches greater value to a product than a lower type, and has a greater marginal willingness to pay for quality. One implication of this assumption is that the reservation price functions of two distinct types cannot cross; another implication is that market segments are closed intervals (cf. Proposition 2.1 below).

There are n firms ($n > 1$) indexed $i = 1, \ldots, n$, each having the same constant (in quantity) unit cost $c(s)$ for a unit of product $s \in \Re_+$. There are no fixed costs. I assume that $c(\cdot)$ is strictly increasing and $u(a, 0) - c(0) \geq 0$. For each consumer type $t \in [a, b]$, I further assume that there exists an optimal quality, $s^*(t)$—the unique solution to the problem $\max_{s \in [0,\infty)}(u(t, s,) - c(s))$—characterized by the local conditions

$$u_s(t, s^*(t)) - c'(s^*(t)) = 0, \tag{2}$$
$$u_{ss}(t, s^*(t)) - c''(s^*(t)) < 0. \tag{3}$$

These assumptions imply that each type (t) of consumer has a distinct favorite product ($s^*(t)$) under marginal cost pricing (cf. Proposition 2.2 below). This property distinguishes this model from Shaked and Sutton [19]. Also, $u(a, 0) - c(0) \geq 0$ implies $u(t, s^*(t)) - c(s^*(t)) > 0$ for all $t \in (a, b]$. Finally, I assume that firms maximize their profits knowing only the distribution of consumers and consumers maximize their surplus perceiving accurately the quality of the product.

Assumption 1 gives a simple structure to the market for each firm (See Fig. 1.)

Proposition 2.1. *Let $s_1 \leq \cdots \leq s_n, s_i \in [s_0, s_{n+1}]$ for $i = 1, \ldots, n$, be n products priced at $p_1, \ldots, p_n$, respectively. Then the market segment for $i (i = 1, \ldots, n)$ defined as*

$$M_i = \{t \in [a, b] : u(t, s_i) - p_i \geq u(t, s_j) - p_j \; for j = 0, 1, \ldots, n + 1\},$$

can be characterized by:

1. *$s_i \leq s_j$ and $p_i > p_j \Rightarrow M_i = \varnothing$,*
2. *$s_i < s_j, t_i \in M_i, t_j \in M_j \Rightarrow t_i \leq t_j$,*
3. *If $M_i \neq \varnothing$, then $M_i = [t_i, t_{i+1}]$, where t_i is either a or the type of consumer indifferent between i. and the next lower quality product with a nonempty market and similarly t_{i+1} is either b or the type of consumer indifferent between i and the next higher quality product with a nonempty market.*

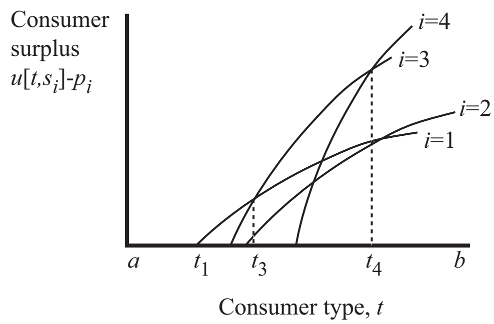

Figure 1: Structure of market segments: Four firms
Firm 1's market segment $= M_1 = [t_1, t_3]$
Firm 2's market segment $= M_2 = \emptyset$
Firm 3's market segment $= M_3 = [t_3, t_4]$
Firm 4's market segment $= M_4 = [t_4, b]$

Proof: Part (1) is obvious. To prove (2) suppose $s_i\, s_j, t_i \in M_i, t_j \in M_j$, and $t_i > t_j$. Then, using $u_{ts} > 0$, $u(t_i, s_j) - u(t_i, s_i) > u(t_j, s_j) - u(t_j, s_i) \geq p_j - p_i$ so that $t_i \notin M_i$, a contradiction. Part (3) follows easily from the definition of market segments and the continuity of u. ■

Another basic feature of the model is that $s^*(a) < s^*(b)$. This immediately implies that any firm, by choosing a price not below marginal cost, can assure itself of positive market share provided its product is located in $[\max(s_0, s^*(a)), s^*(b)]$, and provided the other firms (and the substitutes) are not priced below marginal cost. Indeed, I show later that no firm would find it optimal to choose a product from outside this interval, regardless of the strategies of the other firms.

Proposition 2.2. *$s^*(\cdot)$ is a strictly increasing function. Furthermore, as long as the other firms and the substitute(s) are priced not less than marginal cost, any firm can guarantee for itself a positive market share and positive profits by choosing a distinct product from $[max(s_0, s^*(a)), s^*(b)]$.*

Proof: The monotonicity of s* is obtained by totally differentiating (2) with respect to t and using (1) and (3). As for the second part we note that since s* is strictly increasing, it has an inverse function t* defined by $s^*(t^*(s_i)) = s_i$ for $i = 1, \ldots, n$, and

$$u(t^*(s_i), s_i) - c(s_i) \geq u(t^*(s_i), s_j) - c(s_j) \qquad (\text{ by the definition of } s^*(t^*(s_i)))$$

$$\geq u(t^*(s_i), s_j) - p_j, \qquad j = 0, 1, \ldots, n+1.$$

Hence, by pricing just above marginal cost, firm i can assure itself of $t^*(s_i)$ and a positive interval surrounding it while earning a positive profit. ■

Henceforth, I shall denote $\max(s_0, s^*(a))$ by $\underline{s}$ and $\min(s_{n+1}, s^*(b))$ by $\overline{s}$. Although Proposition 2.2 assures us that no firm would be priced out of the market in any equilibrium if $s_0 < s^*(b)$ in the one-substitute case and $s_0 < \overline{s}$ in the two-substitutes case, it does not preclude the possibility that either substitute will be priced out. This will certainly happen if p_o is greater than $u(b, s_0)$ or p_{n+1} is greater than $u(b, s_{n+1})$. But for $c(s_0) \le p_0 \le u(a, s_0)$ and $c(s_{n+1}) \le p_{n+1} \le u(b, s_{n+1})$, it depends upon the locations of s_0 and s_{n+1}, how large p_0 and p_{n+1} are, and n itself. In the one-substitute case with given s_0 and n, for example, one can imagine that there exists a p_0^* such that for $p_0 < p_0^*$ the Cournot-Nash equilibrium gives a positive market share to the lower substitute, and for $p_0 \ge p_0^*$ the lower substitute is priced out in equilibrium (with $p_0 = p_0^*$ yielding the corner solution). And in the two-substitutes case, when the upper substitute is priced out we are in the one-substitute case. In the analysis that follows I shall be primarily concentrating on the equilibria that assign positive market share to the substitutes. This is because the cases where the substitutes are priced out in equilibrium admit a multitude of Cournot-Nash equilibria where each firm's quantity choice (and hence its quality choice) is essentially determined by the quantity choices of the other firms; see Section 4. (In other words, the "market covered" equilibrium is uninteresting in the Cournot model, unlike the Bertrand model; cf. Gabszewicz and Thisse [7,8].) One set of sufficient conditions for assuring positive market share to the substitutes is $s_0 = s^*(a), p_0 = c(s_0), s_{n+1} = s^*(b)$, and $p_{n+1} = c(s_{n+1})$.

To close this section, let me note one more thing. The assumption of a uniform distribution of consumer types is made with no loss of generality so far. This is because all the properties assumed for $u(\cdot, \cdot)$ so far have involved only the ordering of the t's; thus, an arbitrary distribution of types, say $H(\cdot)$, can be converted to a uniform distribution by transforming the type index to $t' = H(t)$ without losing those properties. But this transformation does affect u_{tt} and possibly higher-order derivatives based on u_{tt}. Later in the paper, when we study the properties of the Cournot-Nash equilibrium, we will implicitly make assumptions on u_{tt} while assuming a uniform distribution. For these properties to generalize beyond the uniform distribution, the t used in this paper must be interpreted as the transformed type index obtained by converting an arbitrary distribution of types into a uniform distribution. Similarly, assumption (3) restricts transformations on the quality index.

3 Cournot-Nash Equilibrium

Let $P \subset \Re_+$ be the (compact) feasible set of products for each firm. Each firm $i(i = 1, \ldots, n, n > 1)$ chooses a product-quantity pair, $(s_i, q_i) \in P \times [0, 1]$, and conjectures the product-quantity choices, (s_{-i}, q_{-i}) of the other n-1 firms; of course, we require of conjectures that $s_j \in P$ for $j \neq i$ and $\sum_{j\neq i} q_j \in [0,1]$[3] The "quantities" of substitutes, q_0 and q_{n+1}, are obtained endogenously as residual market shares. One obvious requirement for feasible quantity choices with a given conjecture is $q_i \in [0, 1 - \sum_{j\neq i} q_j]$ for $i = 1, \ldots, n$. Another reasonable requirement is that each firm be able to compute a set of market-clearing prices for the chosen and conjectured products.

Definition 3.1. *$p \in \Re^n_+$, is a* market-clearing price vector *for $q \in [0,1]^n$ ($\sum_{i=1}^n q_i \in [0,1]$) under $s \in P^n$ if the market shares $m_i(s,p)$ induced by (s,p) satisfy $m_i = \sum_{sj=si} q_j$ for $i = 1, \ldots, n$.*

Note that if two firms have the same product then they have the same induced market share—the sum of their chosen quantities.

Definition 3.2. *$(s,q) \in P^n \times [0,1]^n$ is feasible if $\sum_{i=1}^n q_i \in [0,1]$ and there exists a market-clearing price vector for q under s.*

Not all (s,q) are feasible; for example, if $u(t,s) = ts, c(s) = 0.5s^2, a = 0, b = 1$, and n = 2, then $s_1 = 0.2, s_2 = 0.4, q_1 = 0.3$, and $q_2 = 0.6$ is not feasible in the two-substitutes case with substitutes at 0 and 1 priced at marginal cost. Feasible (s,q) do exist, however: Suppose $s_i \in [\underline{s}, \overline{s}]$ for $i = 1, \ldots, n$ and let $m_i > 0 (i = 1, \ldots, n)$ be the respective market shares when each firm prices at marginal cost (cf. Proposition 2.2); then, (s,m) is feasible. How does one compute market-clearing prices? In general, the market-clearing price may not be unique. But if (s,q) is feasible, $q_i > 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n q_i < 1$, then the unique market-clearing price vector for q under s in the two-substitutes case is given by:

$$p_i = p_{i-1} = u(t_i, s_i) - u(t_i, s_{i-1}) \qquad \text{for} \qquad i = 1, \ldots, n+1, \tag{4}$$

where $t_i = b - (b-a)\sum_{s_j \geq s_i} q_j$. (The market shares of the lower and upper substitutes are $q_0 = (t_1 - a)/(b-a)$ and $q_{n+1} = (b - t_{n+1})/(b-a)$ respectively.) To actually compute the market-clearing prices one has to solve for q_0 from (4) given p_0 and $p_{n+1}, q_1, \ldots, q_n$, and substituting $1 - \sum_{i=0}^n q_i$ for q_{n+1}. There exists a unique solution—and hence also a unique market-clearing price vector—because $u_{ts} > 0$. (Put another way, each value of q_0 defines an imputed price for the upper substitute, p'_{n+1}, by (4), and this function, $p'_{n+1}(q_0)$, is strictly increasing because $u_{ts} > 0$. Then, if this function intersects p_{n+1}—the actual price of the upper substitute—it intersects it only once.) For the one-substitute case this computation is simpler because (4) has to hold only for $i = 1, \ldots, n$, *i.e.*, $t_{n+1} \equiv b$ in the one-substitute case. Thus $(s,q) \in P^n \times [0,1]^n$ is always feasible in the one-substitute case as long as $\sum_{i=1}^n q_i \in [0,1]$.

The pricing scheme given by (4) is intuitive. It says that in order to sell a positive quantity of a certain product quality (under given quality–quantity

[3]The quantities are to be interpreted as fractions of the total population of consumers—market shares—and since we have a uniform distribution of consumer types on $[a,b]$, each quantity q can be represented as $(t_2 - t_1)/(b-a)$, where t_2 and t_1 are the upper and lower boundaries of the market segment that consumes q.

conjectures about the other firms), a firm must realize that its "market segment" will be located in the same relative position as its product quality and its price must be such as to make its market segment boundaries indifferent between its product and the "adjacent" product (lower adjacent product for lower market boundary, higher adjacent product for upper market boundary); cf. Proposition 2.1. Equation (4) also tells us that if (s, q) is feasible but $q_i = 0$ for some $i \in \{1, \ldots, n\}$ then the various market-clearing price vectors differ only in the prices of the products with zero supply quantities—essentially one can increase the prices of such products from the prices given by (4) without affecting market-clearing. If (s, q) is feasible and the substitutes have zero market shares under the prices given by (4), then one can lower all n prices by the same amount and still maintain market-clearing. In the former case, when $q_i = 0$ for some $i \in \{1, \ldots, n\}$, all market-clearing price vectors yield the same profits to each firm; in the latter case, when $q_0 = q_{n+1} = 0$, the prices given by (4) yield the maximum profits. So if we adopt the convention of always choosing the price vector (4), continuity of $u(\cdot, \cdot)$ implies that the set of feasible (s, q) is compact, and each firm's profit function is continuous in feasible (s, q)—even at points where $s_i = s_j$ for $i \neq j$. This is a key mathematical difference between the Cournot and Bertrand models.

Definition 3.3. $(s^0, q^0) \in P^n \times [0, 1]^n$ *is a* pure-strategy Cournot–Nash equilibrium *if* (s^0, q^0) *is feasible and, for each* $i = 1, \ldots, n$,

$$\Pi_i(s_i^0, q_i^0; s_{-i}^0, q_{-i}^0) \geq \Pi_i(s_i, q_i; s_{-i}^0, q_{-i}^0)$$

for all $s_i \in P, q_i \in [0, 1 - \Sigma_{j \neq i} q_j^0]$ *such that* $((s_i, s_{-i}^0), (q_i, q_{-i}^0))$ *is feasible.*[4]

In a pure-strategy Cournot-Nash equilibrium no firm has the incentive to change its product-quantity pair unilaterally to another feasible product-quantity pair, and this includes changes in the product ordering. Much of this paper, however, will deal with "local" Cournot-Nash equilibria where the ordering of firms' products is exogenously fixed.

Definition 3.4. $(s^0, q^0) \in P^n \times [0, 1]^n$ *is a* local pure-strategy Cournot–Nash equilibrium *if* (s^0, q^0) *is feasible and for some permutation* $(\sigma(1), \ldots, \sigma(n))$ *of* $(1, \ldots, n)$ *such that* $s_{\sigma(1)}^0 \leq \cdots \leq s_{\sigma(n)}^0$,

$$\Pi_{\sigma(i)}(s_{\sigma(i)}^0, q_{\sigma(i)}^0; s_{-\sigma(i)}^0, q_{-\sigma(i)}^0) \geq \Pi_{\sigma(i)}(s_{\sigma(i)}, q_{\sigma(i)}; s_{-\sigma(i)}^0, q_{-\sigma(i)}^0)$$

for $i = 1, \ldots, n$ *and all* $s_{\sigma(i)} \in [s_{\sigma(i-1)}^0, s_{\sigma(i+1)}^0], q_{\sigma(i)} \in [0, 1 - \sum_{j \neq i} q_{\sigma(j)}^0]$ *such that* $((s_{\sigma(i)}, s_{-\sigma(i)}^0), (q_{\sigma(i)}, q_{-\sigma(i)}^0))$ *is feasible.*

It should be clear that whenever there exists one pure-strategy Cournot-Nash equilibrium there exist $n!$ of them, identical except for the labelling of the firms. This reflects the complete symmetry of the model: the firms are indistinguishable except for their identities. Also, no Cournot-Nash equilibrium

[4] By $((s_i, s_{-i}^0), (q_i, q_{-i}^0))$ we mean $(s_1^0, \ldots, s_{i-1}^0, s_i, s_{i+1}^0, \ldots, s_n^0, q_1^0, \ldots, q_{i-1}^0, q_i, q_{i+1}^0, \ldots, q_n^0)$.

can have any firm making negative profits; after all, a firm can always choose to supply nothing.

As I said in the Introduction, my aim is to compute the local equilibrium and then verify whether it is a global equilibrium. The key to the computability of the local equilibrium is recursivity of the first-order conditions characterizing the local equilibrium. This means, in essence, that for a given ordering of the firms' qualities, $s_1 < \cdots < s_n$, there exists a one-dimensional "statistic," say r, such that knowing the value of r for firm 1, r_1, one can determine the locally optimal values of s_1 and q_1; knowing r_2 also one can determine s_2 and q_2; and, in general, knowing $r_1, r_2, \ldots, r_i$ one can determine the locally optimal values of $(s_j, q_j)_{j \leq i}$.We shall see in the next section that there exists such a statistic in the one-substitute case and it is the upper boundary, t_{i+1}, of each market segment $[t_i, t_{i+1}]$. That is, if firm 1 knows t_2, then it can determine its optimal product and quantity (in the given product ordering), if firm 2 knows t_3 and t_2 it can determine its optimal product and quantity (in the given product ordering), etc. This is because the aspect of higher quality opponents' strategies that is relevant to any firm is their collective supply quantity (not their qualities)—that determines the upper boundary of the firm's market segment (cf. Eq. (4)). Such a statistic does not exist in general in the two-substitutes case; for example, t_2 is not a statistic in general for firm 1 because firm l's market-clearing price is determined in part by the quantities and qualities of all the firms. In an example characterized by $u(t, s) = ts$, however, the two-substitutes case is recursive as well.

4 The One-substitute Case

Fix a product ordering $s_1 < \cdots < s_n$. Consider i's ($i = 1, \ldots, n$) choice of product. The market-clearing price of s_i is given by

$$p_i = p_0 + \sum_{j=1}^{n} (u(t_j, s_j) - u(t_j, s_{j-1})) \qquad \text{for } i = 1, \ldots, n.$$

Then,

$$\begin{aligned} \frac{\partial p_j}{\partial s_i} &= 0, \qquad j < i, \\ &= u_s(t_i, s_i), \qquad j = i, \\ &= -(u_s(t_{i+1}, s_i) - u_s(t_i, s_i)), \qquad j > i. \end{aligned}$$

The effect of an increase in i's product quality, *under Cournot conjectures*, is to increase i's market-clearing price, reduce that of firms upstream ($j > i$), and leave unchanged the prices downstream ($j < i$)—a ripple effect in one direction. Intuitively, i's market-clearing price increases (with s_i) in order to keep t_i—the lower boundary of i's market—indifferent between s_i and s_{i-1} (the market boundaries of all firms being rigid because the quantities supplied do not change). The price charged by $i+1$decreases in order to keep t_{i+1} indifferent between s_{i+1} and s_i—the increase in i's price is not sufficient to keep t_{i+1} from switching to s_i because t_{i+1} is more quality-sensitive than t_i. In contrast, if $q_0 > 0$,

$$\frac{\partial p_j}{\partial q_i} = -(b-a)\left(\sum_{k=1}^{\min(i,j)} (u_t(t_k, s_k) - u_t(t_k, s_{k-1}))\right) < 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

The effect of an increase in the firm's quantity is to reduce all market-clearing prices—a ripple effect in both directions. Note, however, that the price reduction to upstream firms is *due* to the price reductions downstream ($t_{i+1}, \ldots, t_n$ being rigid).

Using these product and quantity derivatives, and assuming that $s_0 \leq s_1$ and $q_0 > 0$ in equilibrium, the first-order conditions characterizing the local equilibrium can be written as

$$u_s(t_i, s_i) - c'(s_i) = 0 \qquad \text{for} \qquad i = 1, \ldots, n \tag{5}$$

and

$$p_i - c(s_i) = q_i(b-a)\left(\sum_{k=1}^{i}(u_t(t_k, s_k) - u_t(t_k, s_{k-1}))\right) \qquad \text{for} \qquad i = 1, \ldots, n. \tag{6}$$

We also have the identities

$$p_i = p_0 + \sum_{k=1}^{i}(u(t_k, s_k) - u(t_k, s_{k-1})) \qquad \text{for} \qquad i = 1, \ldots, n \tag{7}$$

and

$$q_i = (t_{i+1} - t_i)/(b-a) \qquad \text{for} \qquad i = 1, \ldots, n, \tag{8}$$

where $t_{n+1} \equiv b$. The following proposition is an immediate consequence.

Proposition 4.1. *No pure-strategy Cournot-Nash equilibrium will have* $q_i = 0$ *for any* $i = 1, \ldots, n$ *if* $q_0 = 1 - \sum_{i=1}^{n} q_i > 0$.

Proof: Note first that no equilibrium can have $q_i = 0$ for all $i = 1, \ldots, n$. Because if it did, then some firm i can simply choose a distinct quality, s_i', from $(\underline{s}, s^*(b))$ and a positive quantity q_i' such that $s_i' = s^*(t_i')$ (where $t_i' = b-(b-a)q_i'$) and make positive profits:

$$\begin{aligned}
\Pi_i' &= q_i'[p_{i-1} + (u(t_i', s_i') - u(t_i', s_{i-1})) - c(s_i')] \\
&= q_i'[p_{i-1} - c(s_{i-1}) + (u(t_i', s^*(t_i')) - c(s^*(t_i'))) - (u(t_i', s_{i-1}) - c(s_{i-1}))] \\
&> 0 \text{ since } p_{i-1} - c(s_{i-1}) \geq 0 \text{ and } s^*(t_i') \neq s_{i-1}.
\end{aligned}$$

So there must exist some firm j with $q_j > 0$ making positive profits in equilibrium. Now suppose $q_i = 0$ $(i \neq j)$ and $q_0 > 0$ in equilibrium. Then, by choosing the quantity $q_i' = \varepsilon > 0$ and the product $s_i' = s_j$, i can make positive profits, contradicting the definition of an equilibrium. (Observe that if $q_0 = 0$ then the previous construction fails, and it is impossible to rule out $q_i = 0$ in equilibrium.) ∎

Proposition 4.2 notes another basic property, which together with the observation that $q_i > 0$ for $i = 1, \ldots, n$ in equilibrium as long as $q_0 > 0$ means that no two firms choose the same product in equilibria characterized by $q_0 > 0$.

Proposition 4.2. *No pure-strategy Cournot–Nash equilibrium can have $s_i = s_j, q_i > 0$, and $q_j > 0$ for $i \neq j, i, j \in \{0, 1, \ldots, n\}$.*

Proof: Suppose an equilibrium has $s_{i-1} = s_i$ for some i with $q_i > 0$ and $q_{i-1} > 0$, and assume, without loss of generality, that $s_0 \leq \cdots \leq s_{i-1} = s_i < s_{i+1} \leq \cdots s_n$ and $q_i \leq q_{i-1}$.

Note immediately that if $s_0 = s_{i-1}$, then $p_{i-1} \equiv p_0$, and so the equilibrium must have $q_0 = 0$ if $p_0 > c(s_0)$; this contradicts the statement of this proposition. Therefore, if $s_0 = s_{i-1}$, then $p_{i-1} = p_0 = c(s_0) = c(s_{i-1})$ and $i-1$ can make positive profits by choosing a distinct product from $[\underline{s}, s^*(b)]$—a contradiction of the definition of an equilibrium.

Now suppose $s_{i-2} < s_{i-1} = s_i < s_{i+1}$. Let $\underline{t} = b - (b-a)(\sum_{j \geq i-1} q_j)$ and $\overline{t} = b - (b-a)(\sum_{j \geq i} q_j)$. If $s^*(\underline{t}) < s_{i-1}$, then $i-1$ can increase its profits by choosing $s_{i-1} - \varepsilon$ $(\varepsilon > 0)$ by (5) and the concavity of $u(t, \cdot) - c(\cdot)$. (Note that by choosing $s_{i-1} - \varepsilon$, the lower boundary of $i-1$'s market segment remains $\underline{t}$.) If $s^*(\underline{t}) \geq s_{i-1}$, then $s^*(\overline{t}) > s^*(\underline{t}) \geq s_{i-1} = s_i$ (using $q_{i-1} > 0$), and i can increase its profits by choosing $s_i + \varepsilon$. (Note that by choosing $s_i + \varepsilon$, the lower boundary if i's market segment becomes $\overline{t}$.) In either case, we have a contradiction. ∎

Proposition 4.2 does not rule out the possibility of an equilibrium with $p_0 > c(s_0), s_i = s_0$ and $q_0 = 0$. But this requires $s_0 \geq s^*(a)$: if $s_0 < s^*(a)$, then i will choose $s^*(a)$ (by (5)) so s_0 will not be equal to s_i. See also Proposition 4.3 and Remark 1 below.

Proposition 4.3. *Regardless of what the other firms choose, each firm's best response involves the choice of a product from $[\underline{s}, s^*(b)]$ as long as $s_0 < s^*(b)$.*

Proof: Fix a firm i. If $\sum_{j \neq i} q_j = 1$ then $q_i \equiv 0$ and any choice of product yields zero profits to i. On the other hand, for any $q_i > 0$, i maximizes its profits in any product ordering $s_1 < \cdots < s_n$ by choosing $\max(s_0, s^*(t_i))$ as (5) shows, where $t_i = b - (b-a)(\sum_{s_j \geq s_i} q_j) \in [a, b]$. ∎

What if $s_0 \geq s^*(b)$? Then, $s_i = s_0$ for $i = 1, \ldots, n$ and any combination of q_i's such that $\sum_{i=1}^{n} q_i = 1$ is a Cournot-Nash equilibrium—and these are the only equilibria. (Note that this does not violate Proposition 4.2 because $q_0 = 0$ in this equilibrium.) Since I will be mainly studying equilibria in which $q_0 > 0$

it is useful to note a sufficient condition for it. Proposition 4.4 is essentially a corollary of Proposition 2.2.

Proposition 4.4. *If $p_0 = c(s_0)$ and $s_0 \geq s^*(a)$, then every Cournot-Nash equilibrium has $q_0 > 0$.*

Proof: Suppose $p_0 = c(s_0)$, $s_0 \geq s^*(a)$, and $q_0 = 0$ in equilibrium. Then necessarily $s_1 = s_0$ (because if $s_1 > s_0$, then given the equilibrium condition $p_1 - c(s_1) \geq 0, q_0 > 0$ by Proposition 2.2). But if $s_1 = s_0$, then $p_1 = c(s_1)$ and firm 1 can increase its profits by reducing its output (making $q_0 > 0$). ■

Henceforth, I shall assume that the Cournot-Nash equilibria are characterized by $q_0 > 0$. Then, Propositions 4.1 and 4.2 assure us that Eqs. (5)–(8) are necessary conditions for a Cournot–Nash equilibrium. It is clear that these equations have a recursive structure. Thus, (5) for i=1 yields $s_1 = s^*(t_1)$, which when substituted in (6)–(8) for i=1 yields t_2 as a function of t_1, say $f_2(t_1)$, and then (5) for i=2 yields $s_2 = s^*(t_2) = s^*(f_2(t_1))$, which when substituted in (6)–(8) for i=2 yields t_3 as a function of t_1, say $f_3(t_1)$, and so on until (6)–(8) for i=n yields $t_{n+1} = f_{n+1}(t_1)$. But $t_{n+1} \equiv b$ in a local equilibrium, so that we obtain t_1 by solving $f_{n+1}(t_1) = b$, and then, in turn, the products $s_i = s^*(f_i(t_1))$ and the quantities $q_i = (f_{i+1}(t_1) - f_i(t_1))/(b - a)$ for $i = 1, \ldots, n$. More formally, denoting $s^* \circ f_i$ by s_i^*, we can define recursively

$$f_{i+1}(t_1) = f_i(t_1) + \frac{p_0 + \sum_{k=1}^{i}[u(f_k(t_1), s_k^*(t_1)) - u(f_k(t_1), s_{k-1}^*(t_1))] - c(s_i^*(t_1))}{\sum_{k=1}^{i}[u_t(f_k(t_1), s_k^*(t_1)) - u_t(f_k(t_1), s_{k-1}^*(t_1))]} \quad (9)$$

for $i = 1, \ldots, n+1$, where $f_1(t_1) \equiv t_1$ and $s_0^*(t_1) \equiv s_0$, and state the following necessary condition for the existence of a pure-strategy local equilibrium.

Proposition 4.5. *A necessary condition for the existence of a local Cournot-Nash equilibrium in the one-substitute case is that there exist a $t_1 \in [a, b]$ such that $f_{n+1}(t_1) = b$.*

Proof: As discussed above. ■

We can study the asymptotic properties of the local equilibrium using the f_i functions defined by (9). These functions have the property that for $i = 1, \ldots, n$, if $s_1 = s^*(t_1)$, $s_2 = s^*(f_2(t_1)), \ldots, s_{i-1} = s^*(f_{i-1}(t_1))$, $s_{i+1} = s^*(f_{i+1}(t_1))$ and $q_1 = (f_2(t_1) - f_1(t_1))/(b-a), \ldots, q_{i-1} = (f_i(t_1) - f_{i-1}(t_1))/(b-a)$, then $s_i = s^*(f_i(t_1))$ is the best product (and $q_i = (f_{i+1}(t_1) - f_i(t_1))/(b-a)$ the corresponding best quantity) for i in the interval $[s_{i-1}, s_{i+1}]$. At a local equilibrium, then, there exists an imputed product $s_{n+1} = s^*(b)$, which supports this construction. Given this interpretation, it seems reasonable to require that $f_i' > 0$

for $i = 1, \ldots, n+1$. (This assumption holds in the example to be discussed presently.)[5]

Proposition 4.6. *If $f_i' > 0$ for $i = 1, \ldots, n+1$ a local Cournot-Nash equilibrium with $q_0 > 0$, when it exists, is unique. Furthermore, if $(s_1^{n+1}, \ldots, s_{n+1}^{n+1})$ and $(s_1^n, \ldots, s_n^n)$ are the local equilibria corresponding to $n+1$ and n firms, respectively, then $s_1^{n+1} < s_1^n$. In particular, $\lim_{n\to\infty} s_1^n = \underline{s}$.*

Proof: First, uniqueness. Suppose there are two equilibria, one with $t_1, \ldots, t_n$ as the lower market boundaries, and the other with $t_1', \ldots, t_n'$ as the lower market boundaries. (These lower boundaries completely characterize the local equilibrium, as Eqs. (5)–(8) show.) Assume $t_1' > t_1$. Then, since $f_2' > 0, t_2' > t_2$, and, in general, $t_{n+1}' = f_{n+1}(t_1') > f_{n+1}(t_1) = t_{n+1}$, but that is impossible since $t_{n+1}' = t_{n+1} = b$ for a local equilibrium.

Now suppose $s_1^{n+1} \geq s_1^n$. Then $t_1^{n+1} \geq t_1^n$ (since $s_1^{n+1} = s^*(t_1^{n+1}), s_1^n = s^*(t_1^n)$, and s^* is strictly increasing), $t_2^{n+1} \geq t_2^n$, and, in general, $t_i^{n+1} \geq t_i^n$ for $i = 1, \ldots, n+1$. In particular, $t_{n+1}^{n+1} \geq t_{n+1}^n = 1$, and then $t_{n+2}^{n+1} > 1$, which is impossible since $(s_1^{n+1}, \ldots, s_{n+1}^{n+1})$ is a local pure-strategy Cournot–Nash equilibrium with n + 1 firms. Hence $s_1^{n+1} < s_1^n$.

Finally, since s_1^n is strictly decreasing in n and bounded below by $\underline{s}$ (by (5)), $\lim_{n\to\infty} s_1^n$ exists and is equal to $\underline{s}$.

■

Corollary 4.1. *If $f_i' > 0$ for $i = 1, 2, \ldots, \infty$ and a local Cournot-Nash equilibrium with $q_0 > 0$ exists for $n = 1, 2, \ldots, \infty$, then $\lim_{n\to\infty} t_1^n = \underline{t} = \max(t^*(s_0), a)$.*[6]

Note the requirement in Proposition 4.6 that $q_0 > 0$ in the local equilibria. If $q_0 = 0$, then, in general, there exists a continuum of equilibria characterized by different divisions of the market among the n firms; see Remark 2 below.

Theorem 4.1. *Suppose $f_i' > 0$ for $i = 1, 2, \ldots, \infty$, and a local Cournot-Nash equilibrium with $q_0 > 0$ exists for all n. Then:*

1. *As $n \to \infty$, $\{t_1^n, \ldots, t_n^n\} \to T = [\underline{t}, b]$ and $\{s_1^n, \ldots, s_n^n\} \to S = [\underline{s}, s^*(b)]$.*
2. *If $q_0^n = \max\{q_0^n, q_1^n, \ldots, q_n^n\}$, then $\{t_1^n, \ldots, t_n^n\}$ becomes uniformly dense in T as $n \to \infty$; if $s^*(\cdot)$ is linear also, then $\{s_1^n, \ldots, s_n^n\}$ becomes uniformly dense in S as $n \to \infty$.*
3. *Under the conditions for (1) above, as $n \to \infty$ the equilibrium becomes efficient if $s_0 \geq s^*(a)$.*

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[5] In what follows I shall denote the number of firms in the oligopoly by a superscript; thus s_1^n denotes the lowest quality product in an n-firm oligopoly. Of course, $s_0^n \equiv s_0$.

[6] $t^*(s_0)$ is defined by $s^*(t^*(s_0)) \equiv s_0$.

Proof: To prove (1), note that every equilibrium must have $\{t_1^n, \ldots, t_n^n\} \subset T$. Also, every t_i^n has a limit as $n \to \infty$ because t_1^n has a limit (Proposition 4.6) and the t_i^n are continuous increasing functions of t_1^n (because $f_i' > 0$). Therefore, as $n \to \infty$ at most a finite number of $(t_i^n - t_{i-1}^n)$ can be bounded above zero; the rest must be arbitrarily close to zero. And these latter firms' profits must also be arbitrarily close to zero because every firm's margin is bounded above by $u(b, s^*(b)) - c(s^*(b))$. Let $B_1 \cdots B_k > 0$ denote $\inf_n(t_{i(1)}^n - t_{i(1)-1}^n), \ldots, \inf_n(t_{i(k)} - t_{i(k)-1})$, respectively, where $\{i(1) < \cdots < i(k)\} \subset \{1, 2, \ldots, \infty\}$ and $t_0^n = \underline{t}$. If $i(1) = 1$, then q_0 is bounded above zero and a firm whose profits are arbitrarily close to zero can increase its profits by increasing its quantity, violating the equilibrium requirement. If $i(1) > 1$, then $t_{i(1)-1}$ and $t_{i(1)-2}$ are arbitrarily close to each other, *i.e.,* $q_{i(1)-2}^n$ is arbitrarily close to zero, whereas $q_{i(1)-1}^n$ is bounded above $B_1/(b-a)$. But this is impossible because if $t_{i(1)-1}^n$ and $t_{i(1)-2}^n$ are arbitrarily close to each other, then $s_{i(1)-1}^n$ and $s_{i(1)-2}^n$ are arbitrarily close to each other (by (5) and the continuous second-order differentiability of u and c). Therefore, it cannot be that some $(t_i^n - t_{i-1}^n)$'s are arbitrarily close to each other while others are bounded above zero. That is, as $n \to \infty$ $(t_i^n - t_{i-1}^n) \to 0$ for $i = 1, 2, \ldots, \infty$. Hence also for the s's.

To prove (2), observe that since $q_0^n = \max\{q_0^n, \ldots, q_n^n\}$, $\sum_{j=0}^n q_j^n = 1$, and $q_0^n \to 0$, the t_i's become equispaced as $n \to \infty$. If $s^*(t)$ is linear, say $dt + e$ $(d > 0)$, then $(s_i^n - s_{i-1}^n) = d(t_i^n - t_{i-1}^n)$ and so a similar result holds for the s's.

Finally, to prove (3), note that the efficient arrangement of n products is obtained by maximizing the total surplus function, $\sum_{i=0}^n \int_{t_i}^{t_{i+1}} (u(t, s_i) - c(s_i))dt$, with respect to (s, p), where t_i is defined implicitly by $u(t_i, s_i) - p_i = u(t_i, s_{i-1}) - p_{i-1}$ for $i = 1, \ldots, n$ and $t_0 \equiv a, t_{n+1} \equiv b$. The first-order conditions defining the efficient arrangement are

$$p_i^e - p_{i-1}^e = c(s_i^e) - c(s_{i-1}^e) \qquad \text{for} \qquad i = 1, \ldots, n, \tag{10}$$

$$\int_{t_i^e}^{t_{i+1}^e} (u_s(t, s_i^e) - c'(s_i^e))dt = 0 \qquad \text{for} \qquad i = 1, \ldots, n. \tag{11}$$

But the equilibrium has $u_s(t_i, s_i) - c'(s_i) = 0$ and $t_{i+1} > t_i$, so that $u_s(t_{i+1}, s_i) - c'(s_i) > 0$. Hence (11) can never be satisfied by an n-firm equilibrium.

However, if $s_0 \geq s^*(a)$, then $(s_{i+1}^n - s_i^n) \to 0$ as $n \to \infty$, for $i = 0, 1, \ldots, n$ (so that $c\left(s_{i+1}^n\right) - c(s_i^n) \to 0$ and $p_i^n - p_{i-1}^n \to 0$ by (7). Therefore, the equilibrium converges to efficiency. (If $s_0 < s^*(a)$, then $p_1 \to p_0 + u(a, s^*(a)) - u(a, s_0) > p_0$.) ■

In Theorem 4.1, results (1) and (3)—asymptotic denseness and asymptotic efficiency—do not require strong conditions: essentially just continuous (second-order) differentiability of the reservation price and cost functions. Result (2) comes in two parts, asymptotic uniform denseness of the t_i's and asymptotic uniform denseness of the s_i's.[7] If s^* is nonlinear, the former may still be true,

[7] Asymptotic uniform denseness is the same as weak convergence to the uniform distribu-

but the latter will not be. The requirement that the quantity of the substitute be the largest may seem rather strong at first but becomes more reasonable when one considers that the price elasticity of supply under Cournot conjectures—the $|\partial p_i/\partial q_i|$'s—goes up as we go up the quality spectrum. The requirement does put restrictions on u_{tt}, because for one thing u_{tt} incorporates the effect of the distribution of consumer types (through the transformation of the type index discussed in Section 1). (And this distribution must have a role in determining the quantities supplied to various segments in equilibrium. If certain type intervals have a lot of consumers in them, then firms serving such intervals will be content with smaller segments in equilibrium.) In the example below, where u is linear in t and s and c is quadratic all of these conditions are satisfied.

4.1 An Example

Let $u(t,s) = ts$ and $c(s) = \alpha s^\beta (\alpha > 0, \beta > 1)$. Then $s^*(b) = (b/\alpha\beta)^{1/(\beta-1)}$ and $s^*(a) = (a/\alpha\beta)^{1/(\beta-1)}$. This example has been widely studied in the literature; see, for example, Mussa and Rosen [16], Gabszewicz and Thisse [7, 8], Shaked and Sutton [19], and Moorthy [15].

The first-order conditions (5)–(6) and the identity (7) become, respectively,

$$t_i = \alpha\beta s_i^{\beta-1} \qquad \text{for} \qquad i = 1, \ldots, n, \tag{12}$$

$$p_i - \alpha s_i^\beta = (t_{i+1} - t_i)(s_i - s_0) \qquad \text{for} \qquad i = 1, \ldots, n, \tag{13}$$

$$p_i = p_{i-1} + t_i(s_i - s_{i-1}) \qquad \text{for} \qquad i = 1, \ldots, n. \tag{14}$$

This reduces to

$$(A/\alpha) + (\beta - 1)(s_1^\beta - s_0^\beta) - \beta s_0(s_1^{\beta-1} - s_0^{\beta-1}) = \beta(s_1 - s_0)(s_2^{\beta-1} - s_1^{\beta-1}), \tag{15}$$

$$(\beta - 1)(s_i^\beta - s_{i-1}^\beta) - \beta s_0(s_i^{\beta-1} - s_{i-1}^{\beta-1}) = \beta(s_i - s_0)(s_{i+1}^{\beta-1} - s_i^{\beta-1}), \tag{16}$$

$$(\beta - 1)(s_n^\beta - s_{n-1}^\beta) - \beta s_0(s_n^{\beta-1} - s_{n-1}^{\beta-1}) = \beta(s_n - s_0)(b/\alpha\beta - s_n^{\beta-1}), \tag{17}$$

where $A = p_0 - \alpha s_0^\beta$ and (16) holds for $i = 2, \ldots, n-1$. These equations are nothing but the first-order conditions (13) characterizing the choice of quantity by each firm, with the product choices substituted in. Observe that (15) and (17) are significantly different from the rest: (15) has A in it and (17) has $s^*(b)$ in place of s_{n+1}.

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tion. For example, asymptotic uniform denseness of the equilibrium products can be stated as: The distribution of products, F^n, defined as $F^n(s) = (\#\{s_i^n \leq s\}/n)$, converges weakly to the uniform distribution on S. (A sequence of distribution functions $\langle F^n \rangle$ converges weakly to the distribution function F if $\lim_{n\to\infty} F^n(x) = F(x)$ at the continuity points x of F.)

Products,[a] s_i'		Quantities,[b] q_i'		Profits,[c] Π_i'	
n=2	n=3	n=2	n=3	n=2	n=3
0.5217	0.4371	0.5217α	0.4371α	0.1420α^2	0.0835α^2
0.7826	0.6556	0.4348α	0.3642α	0.1479α^2	0.0870α^2
	0.8337		0.3246α		0.0883α^2
Total surplus $(n = 2) = (0.5520\alpha^2)(s^*(b) - s_0)^3/(b - a)$					
Total surplus $(n = 3) = (0.6160\alpha^2)(s^*(b) - s_0)^3/(b - a)$					

[a] Equilibrium products, $s_i = s_0 + s_i'(s^*(b) - s_0)$.
[b] Equilibrium quantities, $q_i = q_i'(s^*(b) - s_0)/(b - a)$.
[a] Equilibrium profits, $\Pi_i = \Pi_i'(s^*(b) - s_0)^3/(b - a)$

Table 1: The One-Substitute Equilibrium

It is obvious that the system of equations (15)–(17) is recursive. To solve them, one first solves (15) for s_2 as a function of s_1, and then substitutes for s_2 in (16) for $i = 2$ to obtain s_3 as a function of s_1, and so on until (17) becomes an equation in s_1 alone, which when solved gives s_1 and then $s_2, \ldots, s_n$ in turn. The q_i's are then obtained as $(t_{i+1} - t_i)/(b - a)$ from (12).

Remark 1. One thing to keep in mind while solving these equations is that four parameters govern their solution: A, s_0, b, and n. For given n, as A increases the s_1 that solves the system must decrease, ceteris paribus. (For example, for $\beta = 2$ and $n = 2$, if A = 0, the equilibrium has $s_1 - s_0 = (0.5217)s^*(b) - s_0)$, with $A = (0.1\alpha)(s^*(b) - s_0)^2$, $s_1 - s_0 = (0.4160)(s^*(b) - s_0)$, and with $A = (0.15\alpha)(s^*(b) - s_0)^2$, $s_1 - s_0 = (0.3201)(s^*(b) - s_0)$.) Similarly, as b decreases—because $s^*(b)$ decreases—so does s_1, ceteris paribus. Also, as s_0 decreases, so does s_1, ceteris paribus. (And as s_1 decreases, so does s_2, etc.) In particular, for $s_0 < s^*(a)$ the solution to (15)–(17) might yield a $s_1 \leq s^*(a)$, which in turn (via (12)) implies $t_1 \leq a$. But Eqs. (15)–(17) represent the local equilibrium only for $t_1 > a$, *i.e.,* $q_0 > 0$, so that the local equilibrium in these cases must have $q_0 = 0$ and $s_1 = s^*(a)$. Thus, for every A, b, and n there exists a lower bound on s_0, say $\underline{s}(A, b, n)$, such that for $s_0 \leq \underline{s}$, the local equilibrium involves $q_0 = 0, s_1 = s^*(a)$. $\underline{s}(A, b, n)$ is less than $s^*(a)$, increasing in A, decreasing in b, and increasing in n.

Remark 2. It is possible that even though A, s_0, b, and n are such that Eqs. (15)—(17) yield a feasible solution—*i.e.,* $q_0 > 0$—this solution is not a local equilibrium. This will happen if at the solution to (15)–(17), (s_1, q_1), firm 1—the lowest quality firm—prefers to supply $q_1 + q_0$ under quality s_0 rather than q_1 under s_1; cf. Proposition 4.2. This is encouraged by a large A, a large s_0, a small b, and a large n. (For $n = 1$—the monopoly case—for example, if $\beta = 2, s_0 = a/2\alpha$, and $A > (b - a)^2/16\alpha$, then $s_1 = s_0$ and $q_1 = 1$ is the "equilibrium." If $A = (b - a)^2/16\alpha$, then the monopolist is indifferent between this solution and $s_1 = (a + b)/4\alpha, q_1 = \frac{1}{2}$.)

For given A, s_0, and b, if $q_0 = 0$ in an equilibrium with n firms, then that is also the case with any larger number of firms. One sense in which this is true is trivial: we have a series of equilibria characterized by $(m-n) > 0$ firms choosing zero quantity, with the rest supplying the whole market at the qualities and quantities characterizing the n-firm equilibrium. For example, in the example just considered, if $A > (b-a)^2/16\alpha$, then $s_1 = s_0, q_1 = 1, s_i \in [s_0, \infty), q_i = 0$ for $i = 2, \ldots, n$ is an n-firm equilibrium for any $n > 1$. These equilibria are sustained by the nature of Cournot conjectures; if a firm believes that the other n-1 firms will supply the whole market, then it cannot supply any positive quantity. And the other n-1 firms do want to supply the whole market, because it is an $(n-1)$-firm equilibrium to do so. But there is also a continuum of m-firm equilibria which have all the firms (except the substitute) supplying positive quantities. If n firms are in equilibrium with $a = t_1^n < \cdots < t_n^n$ defining the equilibrium products and quantities, then any reallocation of the market which gives each of the n firms less market share than before and in a manner so that $t_i^{n+1} < t_i^n$ for $i = 2, \ldots, n$ with $a = t_1^{n+1}$ and $q_{n+1}^{n+1} = 1 - \sum_{i=1}^n q_i^{n+1} > 0$ yields a (n+ 1)-firm equilibrium. If A = 0, all equilibria are characterized by $q_0 > 0$, as long as $a < b$ and $s^*(a) \le s_0 < s^*(b)$.

Remark 3. For $A = O$, Eqs. (15)–(17) are also easier to solve. We substitute $s_1 = k_1 s_2, \ldots, s_{n-1} = k_{n-1} s_n$, and $s_n = k_n (1/\alpha\beta)^{1/(\beta-1)}$ and find that

$$k_0 = 0$$
$$k_i = \left(\frac{\beta}{2\beta - 1 - (\beta-1)k_{i-1}^\beta} \right)^{1/(\beta-1)} \qquad \text{for} \qquad i = 1, \ldots, n.$$

Observe that $k_1 = (1/(2\beta-1))^{1/(\beta-1)} \in (0,1)$, and $k_{i-1} \in (0,1)$ implies $k_i \in (0,1)$ for $i = 2, \ldots, n$. Also, since $k_2 > k_1$ and $(\beta/(2\beta-1-(\beta-1)k^\beta))^{1/(\beta-1)}$ is increasing in $k, k_i > k_{i-1}$ for $i = 1, \ldots, n$. Finally, since $t_i = \alpha\beta s_i^{\beta-1}$ and $k_i > 0, f_i' > 0$ for $i = 1, \ldots, n+1$. It is straightforward to verify that $q_0 = \max\{q_1, \ldots, q_n\}$. Theorem 4.1 applies to this example.

Remark 4. In order to check whether a local equilibrium is a Cournot-Nash equilibrium one must check whether any firm would like to change its position in the product ordering assuming that other firms do not change their product or quantity.

For $\beta = 2$, $s_0 \ge s^*(a)$, and $A = 0$, Table 1 gives a Cournot–Nash equilibrium for $n = 2, 3$. For $\beta = 2$, $A = 0$, and $s_0 \ge s^*(a)$ the efficient products are given by $(s_i - s_0)/(s^*(b) - s_0) = 2i/(2n+1)$ for $i = 1, \ldots, n$; note that the equilibrium products are excessively close to $s^*(b)$. However, the total surplus increases with n, as Theorem 4.1 asserts. Furthermore, compared to the Bertrand equilibrium for n = 2—with products chosen before prices—the products are closer together, and each firm is worse off; see Moorthy [15].

5 The Two-substitutes Case

The distinguishing feature of the two-substitutes case is the presence of the upper substitute at s_{n+1}. As a result, each firm must view both boundaries of its market segment as elastic; in the one-substitute case, Cournot conjectures meant that only the lower boundary was elastic.

Fix a product ordering $s_1 < \cdots < s_n$, as in the one-substitute case. The market-clearing price of s_i is given by (4)

$$p_i(s_i, q_i) = p_0 + \sum_{j=1}^{i}(u(t_j, s_j) - u(t_j, s_{j-1})) \qquad \text{for} \qquad i = 1, \ldots, n+1.$$

Now, since $t_j = t_1 + (b-a)\sum_{k=1}^{j-1} q_k$ for $j = 2, \ldots, n+1$, $\partial t_j/\partial s_i = \partial t_1/\partial s_i$ for $j = 1, \ldots, n+1$. Also, since $p_{n+1} = p_0 + \sum_{j=1}^{n+1}(u(t_j, s_j) - u(t_j, s_{j-1}))$,

$$\frac{\partial t_1}{\partial s_i} = \frac{u_s(t_{i+1}, s_i) - u_s(t_i, s_i)}{\sum_{j=1}^{n+1}(u_t(t_j, s_j) - u_t(t_j, s_{j-1}))} > 0 \qquad \text{for} \qquad i = 1, \ldots, n.$$

Let $A_i(s,q) = \sum_{j=1}^{i}(u_t(t_j, s_j) - u_t(t_j, s_{j-1}))$ and $B_i(s,q) = \sum_{j=i+1}^{n+1}(u_t(t_j, s_j) - u_t(t_j, s_{j-1}))$. Then,

$$\begin{aligned}
\frac{\partial p_j}{\partial s_i} &= \left(\frac{A_j}{A_j + B_j}\right)(u_s(t_{i+1}, s_i) - u_s(t_i, s_i)), \qquad j < i, \\
&= \frac{A_i u_s(t_{i+1}, s_i) + B_i u_s(t_i, s_i)}{A_i + B_i}, \qquad j = i, \\
&= -\left(\frac{B_j}{A_j + B_j}\right)(u_s(t_{i+1}, s_i) - u_s(t_i, s_i)), \qquad j > i.
\end{aligned}$$

The effect of an increase in i's product quality, under Cournot conjectures, is to increase its market-clearing price and the prices of products downstream, but to reduce the prices upstream—a ripple effect in both directions. Compare this with the one-substitute case where the firms downstream were unaffected; the difference is solely due to the fact that changes in the prices of the upstream firms affects the market share of the upper substitute, which pushes up or down all markets.

Consider quantity changes by i now. Since $t_j = t_1 + (b-a)\sum_{k=1}^{j-1} q_k$ for $j = 2, \ldots, n+1$, $\partial t_j/\partial q_i = \partial t_1/\partial q_i$ for $j \le i$ and $\partial t_j/\partial q_i = \partial t_1/\partial q_i + (b-a)$ for $j > i$. Then $p_{n+1} = p_0 + \sum_{j=1}^{n+1}(u(t_j, s_j) - u(t_j, s_{j-1}))$ implies $\partial t_1/\partial q_i = -(b-a)B_i/(A_i + B_i) < 0$ for $1, \ldots, n$. Consequently,

$$\begin{aligned}
\frac{\partial p_j}{\partial q_i} &= -(b-a)\left(\frac{B_i}{A_i + B_i}\right)A_j, \qquad j \le i, \\
&= -(b-a)\left(\frac{A_i}{A_i + B_i}\right)B_j, \qquad j > i.
\end{aligned}$$

In order to sell an increased quantity under Cournot conjectures, firm i must reduce its price and anticipate price reductions by all the other firms. In the one-substitute case also all prices were reduced. Finally, note that the one-substitute case corresponds to $B_i \equiv \infty$.

Assuming that q_0, $q_{n+1} > 0$, the first-order conditions characterizing the local equilibria are

$$\left(\frac{A_i}{A_i + B_i}\right) u_s(t_{i+1}, s_i) + \left(\frac{B_i}{A_i + B_i}\right) u_s(t_i, s_i) - c'(s_i) = 0 \tag{18}$$

and

$$p_i - c(s_i) = q_i(b-a)\left(\frac{A_i B_i}{A_i + B_i}\right) \qquad \text{for} \qquad i = 1, \ldots, n. \tag{19}$$

And we also have the identities

$$p_i = p_0 + \sum_{j=1}^{i} (u(t_j, s_{j-1}) - u(t_j, s_{j-1})) \qquad \text{for} \qquad i = 1, \ldots, n+1 \tag{20}$$

and

$$q_i = (t_{i+1} - t_i)/(b-a) \qquad \text{for} \qquad i = 1, \ldots, n \tag{21}$$

It is clear that if $u(t, s)$ is nonlinear in t, $A_i + B_i$ is a function of all products and all quantities and the system (18)–(21) cannot be recursive.

5.1 Back to the Example

Since $u(t, s) = ts$, $A_i = s_i - s_0$, $B_i = s_{n+1} - s_i$, and $A_i + B_i = s_{n+1} - s_0$. We show now that we can reduce the necessary conditions for a local equilibrium, (18)–(21), to a recursive system of equations with t_{i+1} (or equivalently s_{i+1}) as a "statistic" that satisfies the properties assumed in Theorem 4.1. And given that Theorem 4.1 did not depend in any essential way upon there being only substitute, it applies here, too.

The conditions (18)–(21) become

$$\left(\frac{s_i - s_0}{s_{n+1} - s_0}\right) t_{i+1} + \left(\frac{s_{n+1} - s_i}{s_{n+1} - s_0}\right) t_i = \alpha\beta s_i^{\beta-1} \qquad \text{for} \qquad i = 1, \ldots, n, \tag{22}$$

$$p_i - \alpha s_i^{\beta} = (t_{i+1} - t_i)\left(\frac{(s_i - s_0)(s_{n+1} - s_i)}{(s_{n+1} - s_0)}\right) \qquad \text{for} \qquad i = 1, \ldots, n, \tag{23}$$

$$p_i = p_{i-1} + t_i(s_i - s_{i-1}) \qquad \text{for} \qquad i = 1, \ldots, n+1. \tag{24}$$

Equations (22)–(24) reduce to

$$\begin{aligned}(s_{n+1} - s_1)(A_0/\alpha + (\beta-1)(s_1^{\beta} - s_0^{\beta}) - \beta s_0(s_1^{\beta-1} - s_0^{\beta-1})\\ = (s_1 - s_0)(\beta s_{n+1}(s_2^{\beta-1} - s_1^{\beta-1}) - (\beta-1)(s_2^{\beta} - s_1^{\beta})),\end{aligned} \tag{25}$$

$$(s_{n+1} - s_i)((\beta - 1)(s_i^\beta - s_{i-1}^\beta) - \beta s_0(s_i^{\beta-1} - s_{i-1}^{\beta-1}))$$
$$= (s_i - s_0)(\beta s_{n+1}(s_{i+1}^{\beta-1} - s_i^{\beta-1}) - (\beta - 1)(s_{i+1}^\beta - s_i^\beta)), \tag{26}$$

$$(s_{n+1} - s_n)((\beta - 1)(s_n^\beta - s_{n-1}^\beta) - \beta s_0(s_n^{\beta-1} - s_{n-1}^{\beta-1})$$
$$= (s_n - s_0)(A_{n+1}/\alpha + \beta s_{n+1}(s_{n+1}^{\beta-1} - s_n^{\beta-1}) - (\beta - 1)(s_{n+1}^\beta - s_n^\beta)), \tag{27}$$

where $A_0 = p_0 - \alpha s_0^\beta$, $A_{n+1} = p_{n+1} - \alpha s_{n+1}^\beta$, and (26) holds for $i = 2, \dots, n-1$. These equations, as in the one-substitute case, are nothing but the first-order conditions (23) (divided by α) characterizing the choice of quantity by each firm. These equations are recursive, although, unlike the one-substitute case, s_i is a *nonlinear* function of s_{i+1}.

Remark 5. For $\beta = 2$, $A_0 = A_{n+1}$, and s_0, $s_{n+1} \in [s^*(a), s^*(b)]$, the solution to (25)–(27) is symmetric: $s_1 - s_0 = s_{n+1} - s_n$, $s_2 - s_1 = s_n - s_{n-1}$, etc. This is a key feature of the two-substitutes case. It is straightforward (but extremely tedious) to verify that the conditions of Theorem 4.1 are satisfied.

Remark 6. For $n = 2$, $\beta = 2$, s_0, $s_{n+1} \in [s^*(a), s^*(b)]$, and $A_0 = A_{n+1} = A$, s_1 is given by a cubic equation. With A expressed as $k\alpha(s_3 - s_0)^2$, the cubic equation yields a feasible solution for $k \le 0.1101$. For $k = 0$, $s_1 - s_0 = 0.382(s_3 - s_0)$; for $k = 0.05$, $s_1 - s_0 = 0.3392(s_3 - s_0)$; for $k = 0.1$, $s_1 - s_0 = 0.2629(s_3 - s_0)$; and for $k = 0.1101$, $s_1 - s_0 = 0.2088(s_3 - s_0)$ is the local equilibrium. As in the one-substitute case, s_1 decreases as A increases. If $k > 0.1101$, the equilibria are characterized by $q_0 = q_{n+1} = 0$. See Remark 2.

For $n = 2, 3$, $\beta = 2$, $A_0 = A_{n+1} = 0$, s_0, $s_{n+1} \in [s^*(a), s^*(b)]$, Table 2 gives the Cournot-Nash equilibrium. Note that the equilibrium is symmetric and profits increase toward the center. Again, for $n = 2$, each firm makes lower profits than in the corresponding Bertrand equilibrium; and again the products are closer together. The efficient product locations are given by $(s_i - s_0)/(s_{n+1} - s_0) = i/(n + 1)$ for $i = 1, \dots, n$; note that the equilibrium products are too far away from the substitutes. However, the total surplus increases with n.

6 Conclusion

In this paper I studied product and quantity competition in a static Cournot framework. Each firm chose a product quality and a supply quantity, simultaneously. I identified the necessary conditions for a Nash equilibrium and showed in an example that there exists one. The interesting thing about this result is that there exists no pure-strategy Nash equilibrium in product-price pairs. The chief mathematical feature of the Cournot model that leads to the existence result is the continuity of each firm's profit function with respect to the firms' strategies. (The corresponding Bertrand model lacks continuity.) Furthermore, in the case where there is only one substitute, the Cournot equilibrium can be computed easily because the first-order conditions defining the equilibrium have a recursive structure. (The Bertrand equilibrium with firms choosing products before prices cannot be computed easily because the first-order conditions defining the "product equilibrium" do not have a recursive structure.) This recursive structure is of course a direct consequence of the nature of consumer preferences in my model—consumer types can be ordered on their marginal willingness to pay for quality.

I also studied the role of the substitutes in the equilibrium. In general, as the prices of the substitutes increase, the market shares of the substitutes decrease in equilibrium. And when the substitutes are priced out in an equilibrium with n firms, they are also priced out in equilibria with more than n firms. This does not necessarily impose an upper bound on the number of firms that can have positive market share in equilibrium—as it does in Shaked and Sutton [19]—because although it obviously is an $(n+1)$-firms equilibrium for the $(n+1)$th firm to choose zero quantity when the other n firms supply the whole market in an n-firm equilibrium, it is not the only $(n+1)$-firms equilibrium. There are equilibria for any n, where each firm chooses a distinct product and a positive market share, and has positive profits.

My principal results established conditions for the asymptotic efficiency of the equilibria. (The equilibria are inefficient for any finite n.) These conditions were satisfied in the example. More interestingly, perhaps, as $n \to \infty$ the set of equilibrium products becomes uniformly dense on the efficient interval of products or the interval defined by the substitutes, whichever is smaller. Asymptotic denseness of the equilibrium products is easily achieved—essentially all that is required is continuous (second-order) differentiability of the reservation price and marginal cost functions. Asymptotic uniform denseness, however, requires stronger conditions. For a uniform distribution of consumer types, I summarized these (sufficient) conditions as the requirement that the substitute's quantity be the largest among all firms. In turn, this property depends upon the reservation price function (especially its variation with respect to the type parameter) and the marginal cost function. In the example, where reservation prices were linear in type and quality and the marginal cost function was quadratic, it was satisfied.

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Products,[a] s'_i		Quantities,[b] q'_i		Profits,[c] Π'_i	
n=2	n=3	n=2	n=3	n=2	n=3
0.382	0.32	0.382α	0.32α	$0.0344\alpha^2$	$0.0218\alpha^2$
0.618	0.50	0.382α	0.30α	$0.0344\alpha^2$	$0.0224\alpha^2$
	0.68		0.32α		$0.0218\alpha^2$
Total surplus $(n = 2) = (0.6304\alpha^2)(s_3 - s_0)^3/(b - a)$					
Total surplus $(n = 3) = (0.6392\alpha^2)(s_4 - s_0)^3/(b - a)$					

[a] Equilibrium products, $s_i = s_0 + s'_i(s_{n+1} - s_0)$.
[b] Equilibrium quantities, $q_i = q'_i(s_{n+1} - s_0)/(b - a)$.
[a] Equilibrium profits, $\Pi_i = \Pi'_i(s_{n+1} - s_0)^3/(b - a)$

Table 2: The Two-Substitutes Equilibrium