REPUTATION IN REPEATED SECOND-PRICE AUCTIONS

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Abstract

A model in which two bidders take part in a series of second-price, common-value, auctions is examined. The question of an optimal auction from an auctioneer's standpoint, in a repeated auction setting, is partially addressed. It is shown that the results from single auction models do not carry over to repeated auctions, when one of the bidders is endowed with a reputation for bidding aggressively. Second-price auctions with two bidders are highly susceptible to manipulative behavior by an aggressive bidder, and yield much lower revenues to the auctioneer.

1 Introduction

Often, the same bidders take part in a series of similar but independent auctions over a period of time. A handful of oil companies participate in auctions of oil leases conducted by the U.S. Department of Interior; a few companies bid for contracts to supply electric power equipment; defense contractors bid for the development of weapons systems. In such situations, each bidder can draw inferences about the others from their past behavior. Therefore, in computing optimal strategies in any auction, bidders take into account not only the history, but also the effect of their strategy on the strategies of others in subsequent auctions, even though these auctions may be independent. This paper examines a repeated auctions model with incomplete information to capture some of these effects. In particular, the following question is addressed: Does it pay any of the bidders to establish or maintain a reputation for bidding aggressively, since this intensifies the "winner's curse" for the other bidder and forces him to submit lower bids in subsequent auctions?

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In addition, the design of an optimal auction from the seller's stand-point is partially addressed. Milgrom and Weber [9] show that in a single common-value auction, the seller prefers a second-price auction to a first-price auction. However, as shown here, this result does not carry over to a repeated auction model in which a bidder is endowed with a reputation for being aggressive. In fact, second-price auctions are shown to be highly susceptible to such manipulative behavior. A small chance that a bidder may be slightly aggressive results in very low revenues for the seller.

Suppose two risk-neutral bidders take part in a finite series of common-value, second-price, auctions. The true values are distributed independently across different auctions. At the end of an auction, the bids become common knowledge and the next auction is held. Since the two players are symmetric, the symmetric equilibrium is a natural one to play. This involves the two bidders playing their stage game symmetric equilibrium strategies at each stage. Hence, it suffices to look at just one auction as in Ortega-Reichart [10], Wilson [11], and in Milgrom and Weber [9]. Winning conveys information about the true value since it implies that the opponent's bid was lower and hence his signal sufficiently small. This is called the "winner's curse" and must be taken into account in computing the optimal bid.

The situation changes dramatically if there is some incomplete information. Suppose that bidder 2 can be one of two types — the ordinary type whose valuation of the objects in each stage is same as that of bidder 1, or the strong type whose valuations are strictly higher. Bidder 1 assigns a small probability $\delta > 0$ to the event that bidder 2 is of the latter type. δ is common knowledge. Since the strong type can be expected to bid higher, δ can be thought of as the reputation of bidder 2 as an aggressive bidder. This could model a situation where two oil companies bid for oil leases in a series of auctions and one of them is uncertain whether the other has access to a lower cost technology. Alternatively, one bidder might be uncertain about the discount rate used by the other to compute the net present value of the stream of benefits arising from an oil lease.

First, consider this model in a single period framework. Since the game ends after one auction, one might think that bidder 2 has little incentive to hide or advertise his type and that the ordinary bidder 2 will continue to bid as in the complete information model. However, there is an indirect effect due to the winner's curse. Bidder 1 realizes that if he wins, the expected value of the object must be smaller than before since now there is the added possibility that he beat the bidder 2 who values the object more, which implies that bidder 2 may have observed an even lower signal. The winner's curse intensifies for bidder 1 and consequently he submits lower bids. This weakens the winner's curse for the ordinary bidder 2. Therefore, he bids higher, which in turn causes bidder 1 to bid even lower and so on, until a new equilibrium is reached in which bidder 1 submits low bids and bidder 2 submits high bids. The ordinary bidder 2 is better off than before since not only is he more likely to win but he pays a lower price as well, whenever he wins, this being a second-price auction.

In a multi-period set-up, bidder 1 will use his observations of $2's$ bids in previous auctions to draw inferences about his type. If there are many auctions remaining, the ordinary bidder 2 will raise his bids further in order to avoid detection and consequently, bidder 1 will bid even lower. It is shown that in any equilibrium, bidder 1 will not win any of the initial auctions. If bidder 2 is of the ordinary type, bidder 1 may win some of the last few auctions.

Milgrom and Weber [9] show that in a single, common-value auction, a second-price auction is better than a first-price auction, from the seller's point of view. This result does not hold in our model. In a second-price auction, the price paid to the auctioneer is the losing bid. Therefore, in our model, the auctioneer's revenues are much lower than in a complete information version of this game (in which it is common knowledge that bidder 2 is of the ordinary type), because the losing bidder is usually bidder 1, who bids much lower than in the symmetric equilibrium of the complete information game. The results are particularly striking when the true value and the signals are lognormally distributed.[1] In this case, bidder 1 always bids zero in equilibrium and therefore, the auctioneer's revenue is zero. This is independent of the amount of initial uncertainty about bidder $2's$ type (as long as there is some). The only way in which the auctioneer's revenue can be zero in a first-price auction is if all bidders' bid zero. However, this is not an equilibrium. Therefore, if there exists an equilibrium in a first-price auction, the auctioneer's revenue must be positive.

In a common-value, second-price auction, a bidder endowed with a reputation can exploit it to his advantage for two reasons. First, his reputation for aggressive bidding intensifies the winner's curse for the other bidder, forcing him to submit lower bids in equilibrium. Therefore, the aggressive bidder wins more often than if he were without a reputation. Second, the aggressive bidder pays a smaller price whenever he wins, since these are second-price auctions. In a first-price auction the latter effect is absent; maintaining a reputation is costly in the stage game as the aggressive bidder has to pay what he bids, whenever he wins.

The symmetric equilibrium of a symmetric, common-value, second-price auction is used when comparing the auctioneer's revenues with those from other auction mechanisms. However, as shown in Section 3, when there are two bidders, the symmetric equilibrium is unstable under the kind of departure from symmetry considered here. The symmetric equilibrium is not close to any of the equilibria in the game in which there is an arbitrarily small probability that bidder 2 may value the object more by an arbitrarily small amount.

This reputation model is in the spirit of Kreps and Wilson [5] on the chain store paradox, and Kreps, Milgrom, Roberts and Wilson [4] on the repeated prisoners' dilemma. The structure of the equilibrium is quite similar to that in these two papers — the ordinary player 2 imitates the strong player 2 except in the last few stages, when his type may be revealed. However, unlike in the above reputation models, it is not costly in the stage game to maintain a

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[1] This is the usual assumption for oil lease auctions, since the true value is the product of several independent random variables. Therefore, by the central limit theorem, the lognormal distribution is a good approximation.

reputation. At each stage, the expected payoff to the ordinary type of player 2 is greater than his expected payoff in the symmetric equilibrium of a symmetric, common-value, second-price auction.

Fudenberg and Maskin [2] show that in a finitely repeated game with incomplete information, any individually rational payoff of the stage game can be approximately attained as a sequential equilibrium in a game with a large enough number of stages and the right kind of irrationality for one of the players. For each individually rational payoff, they construct a game in which with a small probability one of the players is irrational in a specific way. By varying the type of irrationality, the set of individually rational payoffs can be approximated as a sequential equilibrium payoff in different games. Their paper implies that in looking at finitely repeated games with incomplete information, one should be careful about the possible types of the players. In building a model, only those types which arise naturally should be chosen. There are no irrational types in the model considered here. The strong type of player 2 can be thought of as, say, a more efficient oil firm in an oil lease auction.

The organization of this paper is as follows. The basic model is given in Section 2. In Section 3, the equilibria in the last stage are characterized. It is shown that in equilibrium, bidder 1 will never win in the last stage if bidder 2 is of the strong type. Moreover, the players' equilibrium strategies do not depend on the value of δ, bidder $1's$ probability assessment about bidder $2's$ type, as long as it is positive. Also, the symmetric equilibrium of the symmetric stage game is shown to be unstable under the kind of departure from symmetry considered here. The analysis of the repeated game, is done in Section 4. It turns out that if the number of stages is large enough, then in the earlier stages both types of bidder 2 use the same strategy. Also, bidder 1 never wins an auction except possibly in the last few stages, and then only if bidder 2 is of the ordinary type. In the earlier stages, the equilibrium outcome is unique. The existence of an equilibrium is shown in Section 5. Section 6 concludes this paper.

2 The Model

Two risk-neutral bidders, player 1 and player 2, take part in a sequence of n auctions. In each auction, one object is auctioned and given to the highest bidder. The price paid by the winner is equal to the losing bidder's bid, i.e., these are second-price auctions.[2] Ties are resolved at random. The auctions are indexed backwards—the first auction is called stage $n-1$, since there are $n-1$ auctions after this one, and the last one is called stage 0. The values of the n objects to player 1 are independent and, for simplicity, identically distributed random variables denoted $\tilde{V}^l$, $l = n-1, n-2, ...0$. Player 2 can be one of two types — A or B. Type $2A's$ valuation is also $\tilde{V}^l$, whereas type $2B's$ valuation is

[2]When there are two bidders, a second-price auction is strategically equivalent to an English auction.

$k\tilde{V}^l$, where k is a constant strictly greater than one.[3] k is common knowledge. At each stage l, $\tilde{V}^l$ is unknown to the players. They have a common prior on $\tilde{V}^l$ and each of them gets a signal about $\tilde{V}^l$ before they submit their bids. The players' signals in stage l, denoted $\tilde{X}_1^l$ for player 1 and $\tilde{X}_2^l$ for player 2 of either type, are independent and identically distributed, conditional on the true value. Player 2 knows his own type but player 1 doesn't know player $2's$ type. In the beginning, player 1 assesses a small probability δ^{n-1} that player 2 is of type B. δ^{n-1} is common knowledge. At the end of stage l, the players' bids become common knowledge and this information is used to update δ^l to δ^{l-1}, using Bayes' rule. The values of the objects remain unknown to all the players until the last auction is over. There is no discounting in this model. However, the nature of the results is qualitatively unchanged if either the values of the objects become common knowledge at the end of each stage, or if players discount their payoffs provided the discount factor is not too small.

These rules define a finitely repeated game of incomplete information. We are interested in characterizing equilibria in increasing and continuous pure strategies.[4]

In each stage, the random variables $\tilde{V}$ and $\tilde{X}_i$, $i = 1, 2$ are distributed as follows.[5] It is assumed that $\tilde{V}$ and $\tilde{X}_i$ have density functions. $h(\cdot)$ is the density function of $\tilde{V}$ and its support is $[V^1, V^2]$, with $V^1 \geq 0$, $V^2 > V^1$. The conditional density of $\tilde{X}_i$, $i = 1, 2$, given $\tilde{V} = v$ is $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(\cdot|v)$, and $\mathbf{g}_{\tilde{X}_1|\tilde{V}}(\cdot|v) \equiv \mathbf{g}_{\tilde{X}_2|\tilde{V}}(\cdot|v)$, $\forall v \in [V^1, V^2]$. Also, $\tilde{X}_1$ and $\tilde{X}_2$ are conditionally independent, given $\tilde{V}$. The support of $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(\cdot|v)$ is $\left[\underline{X}(v), \overline{X}(v)\right]$, where $\underline{X} : \left[V^1, V^2\right] \mapsto R$ and $\overline{X} : \left[V^1, V^2\right] \mapsto R$, and $\underline{X}(v) < \overline{X}(v)$, $\forall\ v$. Also, we assume that $\underline{X}(\cdot)$, $\overline{X}(\cdot)$ are either continuous, increasing functions, or are constant. However, for the analysis of the repeated game in Section 4, we will assume that $\underline{X}(\cdot)$, $\underline{X}(\cdot)$ are increasing. The proofs can be modified to include the case when $\underline{X}(\cdot)$ and $\overline{X}(\cdot)$ are constant.

Further, it is assumed that $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(x|v)$ is continuous in x, strictly positive on its support and has the strict monotone likelihood ratio property with respect to v, i.e., if $v' > v$ then, $\frac{\mathbf{g}_{\tilde{X}_i|\tilde{V}}(x|v)}{\mathbf{g}_{\tilde{X}_i|\tilde{V}}(x|v')}$ is a decreasing function of x.

Let,

$$\overline{X}^{-1}(x) \equiv \inf\{v \in [V^1, V^2] : \mathbf{g}_{\tilde{X}_i|\tilde{V}}(x|v) > 0\}.$$

$\overline{X}^{-1}(x)$ is the lowest possible realization of $\tilde{V}$ which is consistent with the signal realization $\tilde{X}_i = x$. Similarly, the highest possible realization of $\tilde{V}$ which

[3] It is enough to assume that $2B's$ valuation, $\tilde{V}_{2b}^l$, is such that $E[\tilde{V}_{2b}^l|\tilde{X}_1^l, \tilde{X}_2^l] > E[\tilde{V}^l|\tilde{X}_1^l, \tilde{X}_2^l]$.

[4] I adopt the following convention: A function f is increasing if $x > y \Longrightarrow f(x) > f(y)$ and non-decreasing if $x > y \Longrightarrow f(x) \geq f(y)$.

[5] Since $\tilde{V}$, $\tilde{X}_i$ are identically distributed across stages, the superscript l on $\tilde{V}^l$ and $\tilde{X}_i^l$ is dropped from now on. The stage will be clear from the context.

is consistent with the signal realization $\tilde{X}_i = x$ is:

$$\underline{X}^{-1}(x) \equiv \sup\{v \in [V^1, V^2] : \mathbf{g}_{\tilde{X}_i|\tilde{V}}(x|v) > 0\}.$$

Given the assumptions on $\underline{X}(\cdot)$ and $\overline{X}(\cdot)$, both $\underline{X}^{-1}(\cdot)$ and $\overline{X}^{-1}(\cdot)$ are non-decreasing and continuous. Let $\underline{\underline{X}} = \underline{X}(V^1)$ be the lowest possible realization of $\tilde{X}_i$ and $\overline{\overline{X}} = \overline{X}(V^2)$ be the highest possible realization of $\tilde{X}_i$.

Let $v(\tilde{X}_1, \tilde{X}_2) \equiv \mathbf{E}[\tilde{V}|\tilde{X}_1, \tilde{X}_2]$ and $v(x, y) \equiv \mathbf{E}[\tilde{V}|\tilde{X}_1 = x, \tilde{X}_2 = y]$, where $\mathbf{E}[\cdot|\cdot]$ is the conditional expectation operator. Since $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(\cdot|v)$ has the strict monotone likelihood ratio property with respect to v, $v(x, y)$ is increasing in both its arguments. See Milgrom [7] for a proof of this result. Also, $v(x, y) = v(y, x)$ as $\tilde{X}_1,\ \tilde{X}_2$ are identically distributed. I further assume that $v(x, y)$ is continuous in both its arguments. This is true for distributions like uniform, lognormal, exponential and others.

Let $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1)$ be the conditional density function of $\tilde{X}_2$ given that $\tilde{X}_1 = x_1$. It can be computed from the density function of $\tilde{V}$ and $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(\cdot|\cdot)$ using Bayes' rule. Since $\mathbf{g}_{\tilde{X}_2|\tilde{V}}(x_2|v)$ is continuous in x_2, $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1)$ is continuous in x_2. $\mathbf{G}_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1)$ denotes the conditional distribution function of $\tilde{X}_2$ given $\tilde{X}_1 = x_1$. The support of $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1)$ is $\left[\underline{X}(\overline{X}^{-1}(x_1)), \overline{X}(\underline{X}^{-1}(x_1))\right]$, since when $\tilde{X}_1 = x_1$, player 1 knows that the realization v of $\tilde{V}$ must belong to the interval $\left[\overline{X}^{-1}(x_1), \underline{X}^{-1}(x_1)\right]$. Therefore, the lowest possible signal player 2 can get is $\underline{X}(\overline{X}^{-1}(x_1))$ and the highest possible signal he can get is $\overline{X}(\underline{X}^{-1}(x_1))$.

Define,

$$\underline{Y}(x) \equiv \underline{X}(\overline{X}^{-1}(x)), \quad \overline{Y}(x) \equiv \overline{X}(\underline{X}^{-1}(x)), \quad \forall x \in [\underline{\underline{X}}, \overline{\overline{X}}].$$

Note that $\underline{Y}(\cdot)$, $\overline{Y}(\cdot)$ are non-decreasing and continuous, and that

$$\underline{Y}(x) < x, \quad \forall x \in (\underline{\underline{X}}, \overline{\overline{X}}]. \tag{2.1}$$

When $\underline{X}(\cdot)$, $\overline{X}(\cdot)$ are increasing functions, $\underline{Y}(\cdot) = \overline{Y}^{-1}(\cdot)$.

In the repeated game, a pure strategy for a player consists of n functions, one for each stage. Each function maps the value of the player's signal and the current value of δ into the player's bid in that stage. The equilibrium concept used is slightly stronger than Harsanyi's *Bayesian Nash equilibrium* (see Harsanyi [3]) in that weakly dominated strategies are not allowed in equilibrium. This rules out certain unreasonable equilibria. Throughout this paper, an equilibrium refers to a Bayesian Nash equilibrium in which none of the strategies is weakly dominated.

It will be shown that as long as $\delta^l > 0$, the equilibrium strategies do not depend on δ^l. Therefore the dependence of the strategies on δ^l will be suppressed to simplify the notation. Let $(S_i^l)_{n-1\geq l\geq 0}$ be a pure strategy of player i, $i = 1, 2A, 2B$, where $S_i^l : [\underline{\underline{X}}, \overline{\overline{X}}] \mapsto R$. Thus player i bids $S_i^l(x)$ in stage l when $\tilde{X}_i = x$. $(S_1^l, S_{2a}^l, S_{2b}^l)_{n-1\geq l\geq 0}$ denotes an equilibrium in pure strategies in the repeated game. I assume that S_i^l are increasing and continuous.

3 The Stage Game

Throughout the rest of this paper, a *symmetric stage game* refers to the stage game with $\delta = 0$. In a symmetric stage game, player 1 knows with probability one that player 2 is of type A; the equilibrium outcomes of this game are the same as in a common-value, second-price auction with two symmetric risk-neutral bidders. Similarly, an *asymmetric stage game* refers to the stage game with $\delta > 0$.

The main result of this section is that in any pure strategy equilibrium in the stage game (that is in the last auction), the probability that player 1 will win against player $2B$ is zero. In equilibrium, player $1's$ bids are lower than the bid player $2B$ would submit if he observed the lowest possible signal consistent with player $1's$ signal. In general, if $(S_1^o, S_{2a}^o, S_{2b}^o)$ is an equilibrium in the asymmetric stage game, (S_1^o, S_{2a}^o) will be an asymmetric equilibrium (in which S_1^o is small), of the symmetric stage game. Furthermore, as shown in Proposition 2, the equilibrium strategies in the stage game do not depend on δ^o, provided $\delta^o > 0$. This simplifies the analysis of the repeated game considerably.

Finally, in Proposition 3 it is shown that the symmetric equilibrium of the symmetric stage game is unstable in the sense that as we let $k \to 1$ in the asymmetric stage game, the symmetric equilibrium is not an element of the limit of the set of equilibria of the asymmetric stage games.

3.1 Analysis of the Stage Game

The following are necessary conditions for $(S_1^o, S_{2a}^o, S_{2b}^o)$ to be an equilibrium in the stage game

$$S_1^o(x) \in \arg\max_p \Big\{ \delta^o \cdot \mathbf{E}\Big[\Big(v(\tilde{X}_1, \tilde{X}_2) - S_{2b}^o(\tilde{X}_2) \Big) 1_{\{S_{2b}^o(\tilde{X}_2) < p\}} \Big| \tilde{X}_1 = x \Big] + (1-\delta^o)\cdot \mathbf{E}\Big[\Big(v(\tilde{X}_1, \tilde{X}_2) - S_{2a}^o(\tilde{X}_2) \Big) 1_{\{S_{2a}^o(\tilde{X}_2) < p\}} \Big| \tilde{X}_1 = x \Big] \Big\}, \quad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}} \Big], \tag{3.1}$$

$$S_{2a}^o(x) \in \arg\max_p \mathbf{E}\Big[\Big(v(\tilde{X}_1, \tilde{X}_2) - S_1^o(\tilde{X}_1) \Big) 1_{\{S_1^o(\tilde{X}_1) < p\}} \Big| \tilde{X}_2 = x \Big], \quad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}} \Big], \tag{3.2}$$

$$S_{2b}^o(x) \in \arg\max_p \mathbf{E}\Big[\Big(kv(\tilde{X}_1, \tilde{X}_2) - S_1^o(\tilde{X}_1) \Big) 1_{\{S_1^o(\tilde{X}_1) < p\}} \Big| \tilde{X}_2 = x \Big], \quad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}} \Big], \tag{3.3}$$

where $1_{\{F\}}$ denotes the indicator function of an event F. The expression on the right hand side of (3.1), say, is the expected profit of player 1, when $\tilde{X}_1 = x$ and he bids p. The value of p which maximizes his expected profit is $S_1^o(x)$.

Clearly, bids outside the support of the distribution of the true value, conditional on the signal, are weakly dominated. Since, by assumption, equilibrium strategies are not weakly dominated and the support of the conditional distribution of the true value, $\tilde{V}$, given that $\tilde{X}_i = x$, is $[\overline{X}^{-1}(x), \underline{X}^{-1}(x)]$, we have,

$$\overline{X}^{-1}(x) \le S_1^o(x), S_{2a}^o(x) \le \underline{X}^{-1}(x), \qquad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right], \tag{3.4}$$

$$k\overline{X}^{-1}(x) \le S_{2b}^o(x) \le k\underline{X}^{-1}(x), \qquad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]. \tag{3.5}$$

Next, let $L_{1a}^l(x)$ denote the lowest possible signal that player 1 can get and still win against player $2A$ in stage l when $\tilde{X}_2 = x$ and the strategies $(S_1^l, S_{2a}^l, S_{2b}^l)$ are being played, i.e.,

$$L_{1a}^l(x) \equiv \inf\{y \in [\underline{\underline{X}}, \overline{\overline{X}}] : S_1^l(y) > S_{2a}^l(x)\}, \quad l = n-1, n-2, ...0.$$

If the infimum is over an empty set, define $L_{1a}^l(x) \equiv \overline{\overline{X}}$. Otherwise $L_{1a}^l(x) = (S_1^l)^{-1}(s_{2a}^l(x))$ and $S_1^l(L_{1a}^l(x)) = S_{2a}^l(x)$, since the strategies are increasing and continuous. The dependence of L_{1a}^l on S_1^l and S_{2a}^l is suppressed in the notation. Similarly, let

$$L_{1b}^l(x) \equiv \inf\{y \in [\underline{\underline{X}}, \overline{\overline{X}}] : S_1^l(y) > S_{2b}^l(x)\}, \quad l = n-1, n-2, ...0;$$

$$L_{2a}^l(x) \equiv \inf\{y \in [\underline{\underline{X}}, \overline{\overline{X}}] : S_{2a}^l(y) > S_1^l(x)\}, \quad l = n-1, n-2, ...0;$$

$$L_{2b}^l(x) \equiv \inf\{y \in [\underline{\underline{X}}, \overline{\overline{X}}] : S_{2b}^l(y) > S_1^l(x)\}, \quad l = n-1, n-2, ...0;$$

where $L_{1b}^l(x)$ is the lowest signal player 1 can get and still win against player $2B$ when $\tilde{X}_2 = x$ in stage l; $L_{2a}^l(x)$ is the lowest signal player $2A$ can get and still win against player 1 when $\tilde{X}_1 = x$ in stage l; $L_{2b}^l(x)$ is the lowest signal player $2B$ can get and still win against player 1 when $\tilde{X}_1 = x$ in stage l.

Before analyzing the asymmetric stage game, we need to examine the equilibria of a symmetric stage game. Let S_{2a}^o be $2A's$ best reply to any increasing strategy S_1^o played by bidder 1 in a symmetric stage game. Suppose y_0 is some realization of $\tilde{X}_2$ and that $x_0 = L_{1a}^o(y_0)$. Define $v_{y_0}(x) \equiv v(x, y_0), \forall x$. Then for a small interval around x_0, $v_{y_0}(x)$ must be as shown in Fig. 1, i.e., for some $\epsilon_0 > 0$,

$$v_{y_0}(x) \le S_1^o(x), \quad \forall x \in [x_0, x_0 + \epsilon_0], \tag{3.6}$$

$$v_{y_0}(x) \ge S_1^o(x), \quad \forall x \in [x_0 - \epsilon_0, x_0]. \tag{3.7}$$

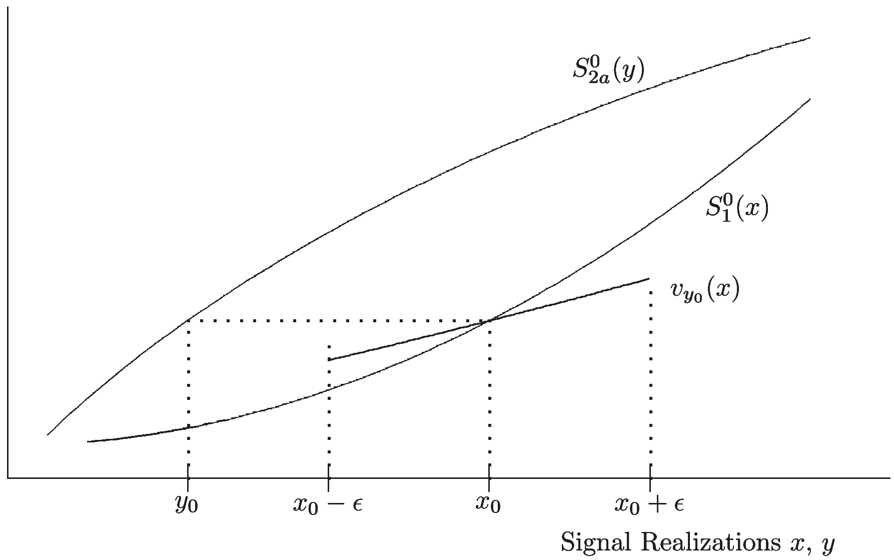

FIGURE 1

These are necessary conditions for S^o_{2a} to be a best reply to S^o_1. To see this, suppose that (3.6) is not true. Then, for all $\epsilon > 0$,

$$v_{y_0}(x_1) > S^o_1(x_1), \quad \text{for some} \;\; x_1 \in [x_0, x_0 + \epsilon]. \tag{3.8}$$

Let $y_1 = L^o_{2a}(x_1)$, where $x_1 \geq x_0$ satisfies (3.8) for some $\epsilon > 0$. Since $y_1 \geq y_0$, the monotone likelihood ratio property implies that $v_{y_1}(x_1) \geq v_{y_0}(x_1)$. Hence, $v_{y_1}(x_1) > S^o_1(x_1)$. The continuity of $v_{y_1}(\cdot)$, $S^o_1(\cdot)$ implies that there exists $\epsilon_1 > 0$ such that $v_{y_1}(x) > S^o_1(x)$, $\forall x \in [x_1, x_1 + \epsilon_1]$. Therefore, when $\tilde{X}_2 = y_1$, $2A$ is better off beating player 1 whenever $\tilde{X}_1 \in [x_1, x_1 + \epsilon_1]$. Thus bidding $S^o_{2a}(y_1)$ gives him a lower expected payoff than bidding a little higher, which contradicts our assumption that S^o_{2a} is $2A's$ best response to S^o_1. The proof for (3.7) is similar.

Next, take any $x_2 > x_0$. Let $y_2 = L^o_{2a}(x_2)$. Since $y_2 \geq y_0$, the monotone likelihood ratio property implies that $v_{y_0}(\cdot) \leq v_{y_2}(\cdot)$. Then, by the argument used to derive (3.6), we have, for some $\epsilon_2 > 0$,

$$v_{y_0}(x) \leq v_{y_2}(x) \leq S^o_1(x), \quad \forall x \in [x_2, x_2 + \epsilon_2].$$

As the choice of $x_2 > x_0$ was arbitrary we have proved the following

$$v(x, y) \leq S^o_1(x), \quad \forall x \geq L^o_{1a}(y), \; \forall y \in [\underline{\underline{X}}, \overline{\overline{X}}]. \tag{3.9}$$

Similarly, by choosing any $x_3 < x_0$, we can show that[6]

$$v(x,y) \geq S_1^o(x), \quad \forall x \leq L_{1a}^o(y),\ \forall y \in [\underline{\underline{X}}, \overline{\overline{X}}]. \tag{3.10}$$

Milgrom [8] has shown that in a common-value, second-price auction with two bidders with identical valuations, there exist a continuum of asymmetric equilibria. These equilibria are of the form $S_1^o(x) = v(x, h(x))$, $S_2^o(x) = v(h^{-1}(x), x)$, where $h(x)$ is any increasing, surjective function. See also Maskin and Riley [6]. Lemma 1 provides another characterization of the set of equilibria. It is shown that in any symmetric stage game if one player plays a best response to the other player's strategy then the two strategies constitute an equilibrium. The technique of the proof is similar to that in Milgrom [8].

LEMMA 1. *In a symmetric stage game, let S_1^o be any increasing and continuous strategy of player 1 and let S_{2a}^o be player $2A's$ best response to S_1^o. Then S_1^o is player $1's$ best response to S_{2a}^o, i.e., (S_1^o, S_{2a}^o) constitute an equilibrium.*

PROOF: We show below that S_1^o is player $1's$ best response to S_{2a}^o even when player 1 sees $\tilde{X}_2$. Therefore it must be a best response when he doesn't observe $\tilde{X}_2$.

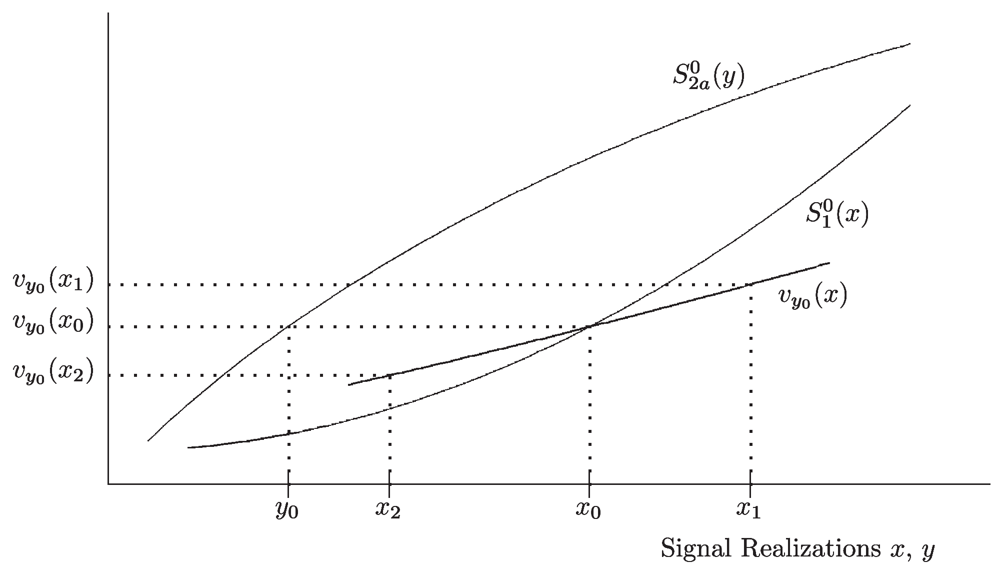

FIGURE 2

Let y_0 be a realization of $\tilde{X}_2$ and let $x_0 = L_{1a}^o(y_0)$. (3.9) and (3.10) imply that the relationship between $S_{2a}^o(y_0)$, $S_1^o(x)$ and $v_{y_0}(x)$ is as shown in Fig. 2.

[6]In deriving (3.9) and (3.10) I have implicitly assumed that $L_{1a}^o(y) \in (\underline{\underline{X}}, \overline{\overline{X}})$. If, for instance, $L_{1a}^o(y) = \underline{\underline{X}}$, then only (3.9) holds.

If $\tilde{X}_1 = x_1$, for any $x_1 > x_0$, then the expected value of the object, conditional on $\tilde{X}_1$ and $\tilde{X}_2$, is $v_{y_0}(x_1)$ which is greater than $2A's$ bid, $S^o_{2a}(y_0)$. Therefore, bidding $S^o_1(x_1)$ is a best response for player 1 since he wins the auction with this bid.

If $\tilde{X}_1 = x_2$, for any $x_2 < x_0$, then the expected value of the object is $v_{y_0}(x_2)$ which is less than $2A's$ bid, $S^o_{2a}(y_0)$. Again $S^o_1(x_2)$ is a best response for player 1 as he loses the auction with this bid.

When $\tilde{X}_1 = x_0$ player 1 is indifferent between winning and losing. ■

In the proof of Lemma 1 the only fact used was that if in a symmetric stage game, S^o_{2a} is player $2A's$ best response to S^o_1, then (3.9) and (3.10) are satisfied. Consider an equilibrium, $(S^o_1, S^o_{2a}, S^o_{2b})$, in an asymmetric stage game. Since S^o_{2a} is $2A's$ best response to S^o_1, (3.9) and (3.10) hold. Therefore S^o_1 must be player $1's$ best response to S^o_{2a} in the *symmetric stage game as well*. Thus, if $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium in an asymmetric stage game then (S^o_1, S^o_{2a}) must be an equilibrium in the symmetric stage game. This is summarized in the following corollary.

COROLLARY 1. *If $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium in any asymmetric stage game, then (S^o_1, S^o_{2a}) is an equilibrium in the symmetric stage game and,* $\forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$,

$$v(x,y) \le S^o_{2a}(y), \quad \forall y \ge L^o_{2a}(x),$$

$$v(x,y) \ge S^o_{2a}(y), \quad \forall y \le L^o_{2a}(x).$$

The next lemma states that $S^o_{2b} \ge S^o_{2a}$. Otherwise, player $2B$ can do better by imitating player $2A$. The proof can be found in Bikhchandani [1].

LEMMA 2. *Let S^o_1 be any strategy of player 1 in the stage game and let S^o_{2a} and S^o_{2b} be the best responses of players' $2A$ and $2B$ respectively, i.e., S^o_{2a} and S^o_{2b} satisfy equations* (3.2) *and* (3.3)*. Then,*

$$S^o_{2b}(x) \ge S^o_{2a}(x), \qquad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right].$$

Moreover, if there is a positive probability that $2B$ will lose to player 1 when $\tilde{X}_2 = x$, i.e., $L^o_{1b}(x) < \overline{Y}(x)$, then

$$S^o_{2b}(x) > S^o_{2a}(x).$$

The main result of this section can now be stated.

PROPOSITION 1. *Suppose that $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium in the asymmetric stage game. Then, if player 2 is of type B, player 1 will never win, i.e, for almost every realization of $\tilde{X}_1$ and $\tilde{X}_2$, $2B$ will win.*

PROOF: Let $\tilde{X}_1 = x_o$, $y_a = L^o_{2a}(x_o)$ and $y_b = L^o_{2b}(x_o)$. Suppose player 1 has a positive probability of winning against $2B$ when $\tilde{X}_1 = x_o$, i.e., $y_b > \underline{Y}(x_o)$, where $\underline{Y}(x_o)$ is the lowest possible realization of $\tilde{X}_2$ when $\tilde{X}_1 = x_o$. Lemma 2 implies that $y_a > y_b$ and therefore by the strict monotone likelihood ratio property, $v(x_o, y_a) > v(x_o, y_b)$. Define,

$$Q(w, y) \equiv \delta^o(v(x_o, w) - S^o_{2b}(w))\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(w|x_o) + (1-\delta^o)(v(x_o, y) - S^o_{2a}(y))\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(y|x_o).$$

Since $v(x, \cdot)$, $S^o_{2a}(\cdot)$, $S^o_{2b}(\cdot)$ and $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(\cdot|x_o)$ are continuous functions, $Q(w, y)$ is a continuous function.

From Corollary 1 we know that

$$v(x_o, y) \leq S^o_{2a}(y), \quad \forall y \geq y_a,$$

$$v(x_o, y) \geq S^o_{2a}(y), \quad \forall y \leq y_a.$$

Therefore $v(x_o, y_a) = S^o_{2a}(y_a)$. Since $S^o_{2a}(y_a) = S^o_{2b}(y_b)$, and $y_a > y_b$, we have $v(x_o, y_b) - S^o_{2b}(y_b) < 0$. Since S^o_{2a}, S^o_1 satisfy (3.4), $y_a \leq \overline{Y}(x_o)$ and, therefore, $y_b < \overline{Y}(x_o)$. Since $y_b > \underline{Y}(x_o)$, and $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(\cdot|x_o)$ is strictly positive on its support, $\mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(y_b|x_o) > 0$. Thus $Q(y_b, y_a) < 0$. The continuity of $Q(w, y)$ implies that there exists $\epsilon > 0$ such that

$$Q(w, y) < 0, \quad \forall w \in [y_b - \epsilon, y_b], \ \forall y \in [y_a - \epsilon, y_a]. \tag{3.11}$$

Comparing (3.11) with (3.1) we see that $S^o_1(x_o)$ cannot be a best response for player 1 when $\tilde{X}_1 = x_o$. Player 1 is strictly better off bidding slightly less. ■

An implication of Proposition 1 is that if $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium in some asymmetric stage game with $\delta = \delta^o$, then it is an equilibrium in any other stage game with $\delta \in [0, 1]$. This is proved in Proposition 2.

PROPOSITION 2. *Suppose that $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium in the asymmetric stage game for some $\delta = \delta^o$, $\delta^o > 0$. Then $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium for any other stage game with $\delta \in [0, 1]$.*

PROOF: Since $(S^o_1, S^o_{2a}, S^o_{2b})$ is an equilibrium for $\delta = \delta^0$, from (3.2) and (3.3) it follows that if bidder 1 plays S^o_1 then the best responses of players $2A$ and $2B$ are S^o_{2a} and S^o_{2b}, regardless of the value of δ. Thus, to complete the proof, I need to show that for all δ, S^o_1 is player $1's$ best response to S^o_{2a} and S^o_{2b}.

From Corollary 1, we know that S^o_1 is player $1's$ best response when $\delta = 0$.

Next, we show that S^o_1 is player $1's$ best response when $\delta = 1$. We know that player $2B's$ best response to S^o_1 is S^o_{2b}. Therefore, an argument similar to that in Lemma 1 establishes that S^o_1 would be player $1's$ best response to S^o_{2b}, if his valuation were $k\tilde{V}$, instead of $\tilde{V}$. However, since player $1's$ valuation is $\tilde{V}$ and not $k\tilde{V}$, his best response to S^o_{2b} must be less than S^o_1. Let $\hat{S}^o_1$ denote his best response. By Proposition 1, player 1 will not win when he uses S^o_1 and player 2 uses S^o_{2b}. Therefore he will not win when he plays $\hat{S}^o_1$ as $\hat{S}^o_1 \leq S^o_1$. His payoff is zero when he plays either S_1^o or $\hat{S}_1^o$. Hence S_1^o is also a best response to S_{2b}^o, i.e., when $\delta = 1$.

Condition (3.1) is satisfied when $\delta = 0$ and $\delta = 1$. Hence, by taking convex combinations of these two extreme values of δ, (3.1) is satisfied for any $\delta \in (0, 1)$. Thus S_1^o is a best response for any $\delta \in [0, 1]$.

It follows that the equilibrium strategies in the stage game do not depend on δ^o. ■

Thus, in every equilibrium in any asymmetric stage game players 1 and $2A$ play as if they were playing in a symmetric stage game; they choose an asymmetric equilibrium of the symmetric stage game in which S_1^o is small enough such that, S_{2b}^o, $2B's$ best response to S_1^o satisfies

$$S_{2b}^o(\underline{Y}(x)) \geq S_1^o(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]. \tag{3.12}$$

A necessary and sufficient condition for S_{2b}^o, S_1^o, to satisfy (3.12) is

$$kv(x, \underline{Y}(x)) \geq S_1^o(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]. \tag{3.13}$$

To show necessity, suppose (3.13) is not satisfied for some x_o, i.e., $kv(x_o, \underline{Y}(x_o)) < S_1^o(x_o)$. Suppose that $\underline{X}(\cdot)$, $\overline{X}(\cdot)$ are constant. Then $\underline{Y}(x) \equiv \underline{\underline{X}}$ and therefore, $L_{2b}^o(x_o) \geq \underline{Y}(x_o)$. Using an argument similar to the one used to derive (3.10), we can show that since S_{2b}^o is a best response to S_1^o, $kv(x_o, L_{2b}^o(x_o)) \geq S_1^o(x_o)$. Therefore, the monotone likelihood ratio property implies that $L_{2b}^o(x_o) > \underline{Y}(x_o)$. Hence (3.12) is not satisfied at x_o.

On the other hand, suppose that (3.13) is not satisfied at x_o, and that $\underline{X}(\cdot)$, $\overline{X}(\cdot)$ are increasing. Therefore, $\underline{Y}(\cdot) = \overline{Y}^{-1}(\cdot)$. Then when $\tilde{X}_2 = y_o \equiv \underline{Y}(x_o)$, $2B$ is better off losing to player 1 whenever $\tilde{X}_1 \in [x_o - \epsilon, x_o]$, for some $\epsilon > 0$. Since $\overline{Y}(y_o) = x_o$, $2B$ is better off bidding $S_1^o(x_o - \epsilon)$, rather than $S_1^o(x_o)$ or higher. Hence (3.12) is not satisfied at x_o.

To prove sufficiency, suppose that S_1^o satisfies (3.13). Then any strategy, S_{2b}^o, which satisfies (3.12) is a best response for $2B$. This is because S_{2b}^o would be a best response for $2B$ even if he knew $\tilde{X}_1$ before submitting his bid, since (3.13) implies $kv(x, y) \geq S_1^o(x)$, $\forall y \in [\underline{Y}(x), \overline{Y}(x)]$, $\forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$.

The equilibrium strategies in the stage game can be computed using Theorem 6.3 in Milgrom [8], which states that in any symmetric stage game if $S_1^o(x) = v(x, h(x))$, and $S_{2a}^o(x) = v(h^{-1}(x), x)$, where $h(x)$ is any increasing, surjective, function, then (S_1^o, S_{2a}^o) is an equilibrium. If we take $h(x)$ small enough such that (3.13) is satisfied, then $(S_1^o, S_{2a}^o, S_{2b}^o)$ is an equilibrium in any stage game with $\delta^o \in [0, 1]$.

COROLLARY 2: *Let $k_1 > k_2 > 1$. If $(S_1^o, S_{2a}^o, S_{2b}^o)$ is an equilibrium in an asymmetric stage game in which $2B's$ valuation is $k_2\tilde{V}$, then it is also an equilibrium in the asymmetric stage game in which $2B's$ valuation is $k_1\tilde{V}$.*

PROOF: Follows directly from the preceding paragraph, and (3.13). ■

We assumed that $\underline{X}(\cdot)$ and $\overline{X}(\cdot)$ are either increasing or constant. Thus all the results of this section hold when $\mathbf{g}_{\tilde{X}_i|\tilde{V}}(\cdot|v)$ is lognormal, exponential, etc. In particular, (3.13) implies that when $\tilde{V},\ \tilde{X}_i|\tilde{V}$ are lognormally distributed player 1 always bids zero. This is because player $2B$ submits bids arbitrarily close to zero when his signal is small enough and since player 1 can never win against $2B$ in equilibrium, he, player 1, must always bid zero.

3.2 Instability of the Symmetric Equilibrium

Propositions 1 and 2 are true regardless of the value of δ and k, as long as $\delta > 0$ and $k > 1$. Also, for a fixed k, the set of equilibria does not depend on $\delta,\ \delta > 0$. Interestingly, as we let $k \to 1$, the limit of the sequence of asymmetric stage games is the symmetric stage game, but the limit of the set of equilibria in the asymmetric stage games does not converge to the set of equilibria in the symmetric stage game. In particular, the limit of the set of equilibria in the asymmetric stage games does not include the symmetric equilibrium of the symmetric stage game. This is shown below.

Let $\{k_r\}_{r=1,2,..}$ be a decreasing sequence of real numbers with $k_r > 1$, and $\lim\limits_{r\to\infty} k_r = 1$. Let Γ_r be an asymmetric stage game in which the valuation of player $2B$ is $k_r\tilde{V}$, and $\delta > 0$. Let E_r^1 be the set of equilibrium strategies of player 1 in Γ_r. That is, if $S_{1,r}^o \in E_r^1$, then there exist $S_{2a,r}^o,\ S_{2b,r}^o$, such that $(S_{1,r}^o, S_{2a,r}^o, S_{2b,r}^o)$ is an equilibrium in Γ_r. Let E^1 be the limit set of the sequence of sets $\{E_r^1\}$, i.e., if $S_1^o \in E^1$ then S_1^o is the limit of a sequence $\{S_{1,r}^o\}$, where $S_{1,r}^o \in E_r^1,\ \forall r$. From Corollary 2, we know that $E_{r+1}^1 \subseteq E_r^1,\ \forall r$. Also, as shown later in Section 5, $\overline{X}^{-1}(x) \in E_r^1,\ \forall r$. Therefore, E^1 is non-empty and is equal to $\cap_{r=1}^{\infty} E_r^1$.

Proposition 3 proves that the symmetric equilibrium strategy does not belong to E^1. As shown in Milgrom [8], and in Milgrom and Weber [9], the symmetric equilibrium strategy in the symmetric stage game is $S_1^o(x) = S_{2a}^o(x) = v(x,x)$.

PROPOSITION 3. $v(x,x) \notin E^1$.

PROOF: Suppose $v(x,x) \in E^1$. Then there exists a sequence of functions $\epsilon_r(x)$, such that $\lim\limits_{r\to\infty} \epsilon_r(x) = 0,\ \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$, and $v(x,x) - \epsilon_r(x) \in E_r^1$. By (3.13),

$$k_r\ v(x, \underline{Y}(x)) \geq v(x,x) - \epsilon_r(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right].$$

Taking the limit as $r \to \infty$, for each x, we have,

$$v(x, \underline{Y}(x)) \geq v(x,x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right].$$

But by (2.1) we know that $x > \underline{Y}(x),\ \forall x \in (\underline{\underline{X}}, \overline{\overline{X}}]$. Therefore, by the strict monotone likelihood ratio property,

$$v(x, \underline{Y}(x)) < v(x, x), \qquad \forall x \in (\underline{\underline{X}}, \overline{\overline{X}}].$$

Contradiction. ■

The symmetric equilibrium of a symmetric, common-value, second-price auction is used when comparing the auctioneer's revenues with those from other auction mechanisms. Proposition 3 shows that when there are two bidders, the symmetric equilibrium is unstable under the kind of departure from symmetry considered here. The symmetric equilibrium is not close to any of the equilibria in asymmetric stage games in which there is an arbitrarily small probability that bidder 2 may value the object more by an arbitrarily small amount. There are asymmetric equilibria which are stable. However, they yield lower revenues to the auctioneer. It would be interesting to investigate the stability of the symmetric equilibrium in a common-value, second-price, auction with more than two bidders.

4 The Repeated Game

In this section, it is shown that the equilibrium in the earlier stages is similar to the one in the last stage, except that player 1 submits even lower bids. The main result of this section is Proposition 4 which states that if the number of stages, n, is large enough, then player 1 will lose all, except some of the last few auctions. The proof is by induction on the number of stages.

4.1 Analysis of the Repeated Game

In analyzing the repeated game, I assume that if at any stage l, player 2 is revealed to be of type A, i.e., $\delta^l = 0$, then $\delta^{l-1} = 0$ and the symmetric equilibrium of the symmetric stage game is played in all subsequent stages. This is because if $\delta^m = 0,\ \forall m \leq l$, then the remaining game after stage l is between two symmetric players. Therefore, the symmetric equilibrium is a natural one to select. In this equilibrium, the players play their symmetric stage game best responses[7] in each stage. As shown in Milgrom [8], the symmetric equilibrium in the symmetric stage game is, $S_1^o(x) = S_{2a}^o(x) = v(x, x),\ \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$.

Also, I assume that $\underline{X}(\cdot),\ \overline{X}(\cdot)$ are increasing, continuous functions. The proofs can be modified to include the case where $\underline{X}(\cdot),\ \overline{X}(\cdot)$ are constant. A final assumption is that $k \in (1, \overline{k})$, for some $\overline{k} > 1$.

Let $\left(\Pi_1^l(\delta^l), \Pi_{2a}^l(\delta^l), \Pi_{2b}^l(\delta^l)\right)_{n-1 \geq l \geq 0}$ be the profit functions associated with an equilibrium, $(S_1^l, S_{2a}^l, S_{2b}^l)_{n-1 \geq l \geq 0}$. $\Pi_i^l(\delta^l)$ is the expected profit for player $i,\ i =$

[7] The stage game best response of a player is his best response when there are no more stages to go.

$1, 2A, 2B$, just before stage l (i.e., before player i sees his signal in stage l but after the bids in stage $l+1$ have become common knowledge), from the rest of the game, including stage l, given that the current value of δ is δ^l and that the above equilibrium is being played.

By Proposition 2, the equilibrium in the stage game, $(S_1^o, S_{2a}^o, S_{2b}^o)$, does not depend on δ^o, provided $\delta^o > 0$. Therefore the profit functions (Π_{2a}^o, Π_{2b}^o) do not depend on δ^o, provided $\delta^o > 0$, and we can define $\Pi_i^o(1) \equiv \Pi_i^o(\delta^o),\ \forall \delta^o \in (0, 1],\ i = 2A, 2B$.

Next, we show that if $k \in (1, \overline{k})$, for some $\overline{k} > 1$, then for all equilibria in an asymmetric stage game, $\Pi_{2a}^o(1) > \Pi_{2a}^o(0)$. First we show that for any $\mu < 1$, there exists $\overline{k} > 1$ such that if $k \in (1, \overline{k})$, then in any equilibrium in the stage game,

$$Pr\{S_1^o(\tilde{X}_1) < v(\tilde{X}_1, \tilde{X}_1)\} \geq \mu. \tag{4.1}$$

Then by choosing μ close enough to 1, we can ensure that $\Pi_{2a}^o(1) > \Pi_{2a}^o(0)$.

Choose $X_l > \underline{\underline{X}}$, $X_u < \overline{\overline{X}}$, such that $Pr\{\tilde{X}_1 \in [X_l, X_u]\} = \mu$. Let $t(x) = \frac{v(x,x)}{v(x, \underline{Y}(x))}$. Since, $\underline{Y}(x),\ v(x, x)$ are continuous, so is $t(x)$.[8] Therefore $t(x)$ attains a minimum on the compact set $[X_l, X_u]$. Let the minimum value be $\overline{k}$. By (2.1), and the strict monotone likelihood ratio property, $t(x) > 1,\ \forall x > \underline{\underline{X}}$. Hence, $\overline{k} > 1$. Therefore, for all $k \in (1, \overline{k}),\ v(x, x) > kv(x, \underline{Y}(x)),\ \forall x \in [X_l, X_u]$. Together with (3.13), this implies that $S_1^o(x) < v(x, x),\ \forall x \in [X_l, X_u]$ and thus (4.1) holds.

Therefore,

$$\begin{aligned}
\Pi_{2a}^o(0) &= \mathbf{E}\left[\left(v(\tilde{X}_1, \tilde{X}_2) - v(\tilde{X}_1, \tilde{X}_1)\right) 1_{\{\tilde{X}_1 < \tilde{X}_2\}}\right] \\
&< \mathbf{E}\left[\left(v(\tilde{X}_1, \tilde{X}_2) - S_1^o(\tilde{X}_1)\right) 1_{\{\tilde{X}_1 < \tilde{X}_2\}}\right] \\
&\leq \mathbf{E}\left[\left(v(\tilde{X}_1, \tilde{X}_2) - S_1^o(\tilde{X}_1)\right) 1_{\{S_1^o(\tilde{X}_1) < S_{2a}^o(\tilde{X}_2)\}}\right] \\
&= \Pi_{2a}^o(1),
\end{aligned}$$

where, since the expectations are bounded, the first inequality follows from (4.1) if we take μ close enough to 1, and the second because S_{2a}^o is a best response to S_1^o. Similarly, $(1 - \delta^o) \cdot \Pi_1^o(0) > \Pi_1^o(\delta^o),\ \forall \delta^o \in (0, 1)$.

Next, suppose that $(S_1^l, S_{2a}^l, S_{2b}^l)_{n-1 \geq l \geq 0}$ is a pure strategy equilibrium in increasing and continuous strategies in the repeated game. At the end of stage l, the bids in that stage are announced and since S_i^l are increasing, the players' signals and the updated value of player $1's$ assessment of player $2's$ type, δ^{l-1}, become common knowledge. If player 2 bids p in stage l and $\tilde{X}_1 = x_1$, then by Bayes' rule,

$$\delta^{l-1}(p) = \frac{\delta^l \cdot \mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(c_{2b}^l(p)|x_1)}{\delta^l \cdot \mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(c_{2b}^l(p)|x_1) + (1 - \delta^l) \cdot \mathbf{g}_{\tilde{X}_2|\tilde{X}_1}(c_{2a}^l(p)|x_1)}, \tag{4.2}$$

[8] Since $V_1 \geq 0$, we have $v(\cdot, \cdot) \geq 0$. If for some x_0, $v(x_0, \underline{Y}(x_0)) = 0$, then we can minimize $t(x)$ on the compact set which remains after removing a small enough open interval around x_0, from $[X_l, X_u]$.

where c_{2a}^l is the inverse of S_{2a}^l, and c_{2b}^l is the inverse of S_{2b}^l. Suppose $\tilde{X}_2 = x_2$. Then if player 2 is of type A he will bid $p = S_{2a}^l(x_2)$ in equilibrium. If he is of type B, then $p = S_{2b}^l(x_2)$ and therefore, $\delta^{l-1} > 0$. Note that δ^{l-1} does not depend on S_1^l. Therefore, player 1 will play his stage game best response to S_{2a}^l, S_{2b}^l in stage l. The dependence of δ^{l-1} on x_1 is suppressed in the notation.

Next, we make the following induction hypothesis.

INDUCTION HYPOTHESIS (IH). *For some integer* $l-1,\ 0 \leq l-1 < n-1$,

$$\Pi_{2a}^{l-1}(\delta^{l-1}) \textit{ and } \Pi_{2b}^{l-1}(\delta^{l-1}) \textit{ are constant } \forall \delta^{l-1} > 0, \tag{4.3}$$

and,

$$\Pi_{2a}^{l-1}(0) < \Pi_{2a}^{l-1}(1), \textit{ where } \Pi_{2a}^{l-1}(1) \equiv \Pi_{2a}^{l-1}(\delta^{l-1}), \quad \forall \delta^{l-1} > 0. \tag{4.4}$$

This hypothesis is true for the last stage, i.e., for $l-1=0$. It will be shown that if (4.3) and (4.4) are satisfied for stage $l-1$, then they are satisfied for stage l and that the equilibrium in stage l is similar to the one in stage 0, except that player 1 bids even lower and players' $2A$, $2B$ bid even higher.

Given that (IH) holds for stage $l-1$, the exact value of δ^{l-1} is no longer required for the computation of the best responses in stage l. We only need to know whether player $2's$ type will be revealed in this stage, i.e., whether δ^{l-1} is zero or positive. Therefore, let $r^l(p,x)$ be the probability assessed by player 2 that he will be revealed to be of type A when he bids p and $\tilde{X}_2 = x$ in stage l, i.e., $r^l(p,x) = Pr\left\{\delta^{l-1}(p) = 0 \middle| \tilde{X}_2 = x\right\}$. Therefore,

$$r^l(p,x) = \begin{cases} 1 - \mathbf{G}_{\tilde{X}_1|\tilde{X}_2}\left(\overline{Y}(c_{2b}^l(p))\middle|x\right), & \text{if } p < S_{2b}^l(x); \\ 0, & \text{if } p = S_{2b}^l(x); \\ \mathbf{G}_{\tilde{X}_1|\tilde{X}_2}\left(\underline{Y}(c_{2b}^l(p))\middle|x\right), & \text{if } p > S_{2b}^l(x). \end{cases} \tag{4.5}$$

If $p \neq S_{2b}^l(x)$, $r^l(p,x)$ is strictly positive and increases[9] as p moves away from $S_{2b}^l(x)$, until $r^l(p,x) = 1$. The shape of $r^l(p,x)$ is shown below.

[9]Here we make use of the assumption that $\overline{X}(\cdot)$ and $\underline{X}(\cdot)$ are increasing.

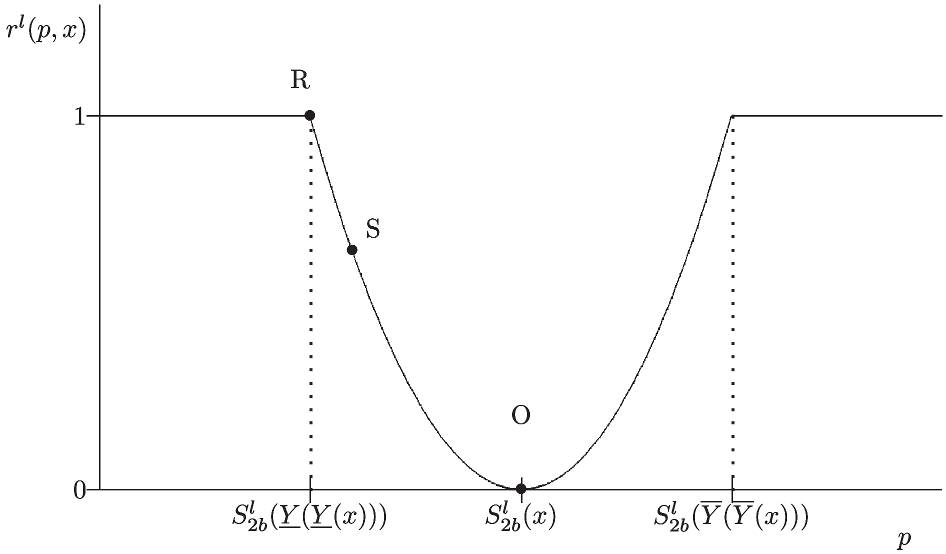

FIGURE 3

Given (4.3) we know that,

$$\mathbf{E}\Big[\Pi_i^{l-1}(\delta^{l-1}(p))|\tilde{X}_2 = x\Big] = r^l(p,x)\cdot\big[\Pi_i^{l-1}(0) - \Pi_i^{l-1}(1)\big] + \Pi_i^{l-1}(1), \quad i = 2A, 2B.$$

This, together with the fact that δ^{l-1} does not depend on player $1's$ strategy in stage l, and that $r^l(s^l_{2b}(x), x)) = 0$, implies that the necessary conditions for $(S^l_1, S^l_{2a}, S^l_{2b})$ to be best responses in stage l are

$$S^l_1(x) \in \arg\max_p \bigg\{ \delta^l \cdot \mathbf{E}\Big[\Big(v(\tilde{X}_1,\tilde{X}_2) - S^l_{2b}(\tilde{X}_2)\Big)\, 1_{\{S^l_{2b}(\tilde{X}_2)<p\}}\Big|\tilde{X}_1 = x\Big] + (1-\delta^l)\cdot\mathbf{E}\Big[\Big(v(\tilde{X}_1,\tilde{X}_2) - S^l_{2a}(\tilde{X}_2)\Big)\, 1_{\{S^l_{2a}(\tilde{X}_2)<p\}}\Big|\tilde{X}_1 = x\Big]\bigg\}, \qquad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}}\Big]. \tag{4.6}$$

$$S^l_{2a}(x) \in \arg\max_p \bigg\{ \mathbf{E}\Big[\Big(v(\tilde{X}_1,\tilde{X}_2) - S^l_1(\tilde{X}_1)\Big)\, 1_{\{S^l_1(\tilde{X}_1)<p\}}\Big|\tilde{X}_2 = x\Big] + r^l(p,x)\cdot\big[\Pi_{2a}^{l-1}(0) - \Pi_{2a}^{l-1}(1)\big]\bigg\}, \quad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}}\Big]. \tag{4.7}$$

$$S^l_{2b}(x) \in \arg\max_p \mathbf{E}\Big[\Big(kv(\tilde{X}_1,\tilde{X}_2) - S^l_1(\tilde{X}_1)\Big)\, 1_{\{S^l_1(\tilde{X}_1)<p\}}\Big|\tilde{X}_2 = x\Big], \quad \forall x \in \Big[\underline{\underline{X}}, \overline{\overline{X}}\Big]. \tag{4.8}$$

Note that (4.6) and (4.8) are identical to (3.1) and (3.3). It is optimal for players 1 and $2B$ to play their stage game best responses in stage l, provided (IH) holds for stage $l-1$.

The next lemma is the repeated game analog of lemma 2. It implies that for all x, $r^l(s^l_{2a}(x),x)$ cannot be to the right of the point O in Fig. 3.

LEMMA 3. *Suppose (IH) holds for stage* $l-1$. *Then in any equilibrium in the repeated game* $S^l_{2b}(x) \geq S^l_{2a}(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$. *Moreover, if there is a positive probability that* $2B$ *will lose to player 1 when* $\tilde{X}_2 = x$, *i.e.,* $L^l_{1b}(x) < \overline{Y}(x)$, *then* $S^l_{2b}(x) > S^l_{2a}(x)$.

Since $\Pi^{l-1}_{2a}(0) < \Pi^{l-1}_{2a}(1)$, player $2A$ has an incentive to hide his type in stage l. From (4.8), we know that S^l_{2b} is player $2B's$ stage game best response to S^l_1, and from Lemma 2 we know that $2B's$ stage game best response is not less than $2A's$ stage game best response. Therefore, $2A's$ best response to S^l_1 in stage l is not less than his stage game best response. The next lemma, which is the repeated game analog of Proposition 1, shows that player 1 cannot win against $2B$ in stage l.

LEMMA 4. *Suppose (IH) holds for stage* $l-1,\ l \geq 1$. *Then in any equilibrium player 1 will not win against* $2B$ *in stage* l.

PROOF: Suppose that for some realization of $\tilde{X}_1$ in stage l, player 1 has a positive probability of winning against $2B$. Then, since S^l_{2a} is at least as high as player $2A's$ stage game best response to S^l_1, the argument in the proof of Proposition 1 implies that S^l_1 is strictly greater than $1's$ stage game best response. But player $1's$ best response in stage l is his stage game best response. Thus we have a contradiction. ■

Comparing (4.6) and (3.1), we know that player $1's$ best response in stage l is his stage game best response. Since, by Lemma 4, he cannot win against player $2B$, S^l_1 must be his stage game best response against S^l_{2a}. Therefore, Lemma 1 implies that S^l_{2a} must be $2A's$ stage game best response, *i.e.,* $2A's$ *best response in stage* l *is his stage game best response to* S^l_1. Therefore either $S^l_{2b} \equiv S^l_{2a}$, or $S^l_{2b}(x) > S^l_{2a}(x)$ for some x, and it is too costly for $2A$ to bid anything higher in order to imitate $2B$. In the former case we have, $r^l(s^l_{2a}(x),x) = 0,\ \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$. In the latter case, $r^l(s^l_{2a}(x),x)$ must be to the left of R in Fig. 3, i.e., $r^l(s^l_{2a}(x),x) = 1$ and $S^l_{2a}(x) < S^l_{2b}(\underline{Y}(\underline{Y}(x)))$. Suppose not. Then $r^l(s^l_{2a}(x),x) \in (0,1)$ for some x, i.e., we are at S in Fig. 3. If $2A$ were to increase his bid slightly he would decrease his chances of being detected in this stage (he would move down towards O along the curve in Fig. 3) and hence increase his chances of getting $\Pi^{l-1}_{2a}(1) - \Pi^{l-1}_{2a}(0)$ more in the continuation game. This results in a first order increase in his payoffs. Although it is costly in the stage game to bid more than $S^l_{2a}(x)$, this has a second order effect on player $1's$ payoffs as $S^l_{2a}(x)$ is his stage game best response to $S^l_1(x)$. Thus $2A$ is better off bidding slightly higher, as the benefits from the continuation game are greater than the costs in the stage game. This contradicts the hypothesis that S^l_{2a} is his best response. Therefore, $r^l(s^l_{2a}(x),x) \in \{0,1\}$. This is the next lemma. The proof can be found in Bikhchandani [1].

LEMMA 5. *Suppose (IH) holds for stage $l-1$. Then,* $r^l(s^l_{2a}(x),x) \in \{0,1\}, \ \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right]$.

When $r^l(s^l_{2a}(x),x) = 1$ and $\tilde{X}_2 = x$, $2A$ is revealed at the end of this stage. $r^l(s^l_{2a}(x),x)$ is to the left of R in Fig. 3. The stage game costs to $2A$ of bidding a lot more than his stage game best response outweigh the benefits in the continuation game of decreasing the probability of being revealed in this stage. The costs of bidding higher than the stage game best response are bounded by $V^2 - V^1$ whereas the benefits from the continuation game, $\Pi^{l-1}_{2a}(1) - \Pi^{l-1}_{2a}(0)$, increase without bound as l increases. Thus when l is large enough $r^l(s^l_{2a}(x),x)$ cannot be one; it must be zero. There exists an integer l^* such that for all $l \geq l^*$, $S^l_{2b} \equiv S^l_{2a}$. This, together with Lemma 4, implies that regardless of player $2's$ type, player 1 will not win in stage l, if $l \geq l^*$. l^* does not depend on δ^{n-1} or k. It depends on which equilibrium is played in the last l^* stages.[10]

In stage l, player 1 will not win against $2B$ and players 1, $2A$ and $2B$ play their stage game best responses, provided (IH) holds for stage $l-1$. Therefore, as in Proposition 2, we can show that S^l_1, S^l_{2a} and S^l_{2b} do not depend on δ^l and that $\Pi^l_{2a}(\delta^l), \ \Pi^l_{2b}(\delta^l)$ are constant for $\delta^l > 0$. Also $\Pi^l_{2a}(\delta^l) > \Pi^l_{2a}(0), \ \forall \delta^l > 0$. Thus, if (IH) holds for stage $l-1$ then it holds for stage l as well. Since (IH) is true for stage 0 we have proved the following:

PROPOSITION 4. *In any pure strategy equilibrium in increasing and continuous strategies,* $(S^l_1, S^l_{2a}, S^l_{2b})_{n-1 \geq l \geq 0}$, *in the repeated game with* $\delta^{n-1} > 0$:
a) All players play their stage game best responses at each stage.
b) There exists an integer l^* *such that,* $S^l_{2b} \equiv S^l_{2a}, \ \forall l \geq l^*$.
Therefore, $\delta^{n-1} = \delta^{n-2} = \ ... \ = \delta^{l^*-1}$.
c) Player 1 will not win in any of the stages if player 2 is of type B. *Hence player 1 will*
never win except possibly in last l^* *stages, and then only if player 2 is of type* A.
d) The equilibrium strategies do not depend on $\delta^l, \ l = 0,1,2,...n-1$.

And, finally:

LEMMA 6. *The equilibrium outcome is unique in the first* $n - l^*$ *stages, and in all equilibria,* $S^l_1(x) = \overline{X}^{-1}(x), \ \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right], \ \forall l \geq l^*$.

PROOF: We know that, regardless of his type, player 2 will win in the first $n-l^*$ stages. The equilibrium outcome in these stages will be unique if player 2 pays

[10] If, instead of the bids the true value becomes common knowledge at the end of each stage, then for $p < S^l_{2b}(x)$, say, $r^l(p,x) = P\{\tilde{V} \in (\ \underline{X}^{-1}(c^l_{2b}(p)), \underline{X}^{-1}(x)] | \tilde{X}_2 = x\}$. The shape of $r^l(p,x)$ remains the same and therefore, the results are essentially unchanged. The same is true if there is discounting, provided the discount factor is not too small. However, the value of l^* will be different.

the same price in all equilibria, i.e., if $S_1^l(x),\ l \geq l^*$, is the same in all equilibria. Hence it suffices to show that in all equilibria,

$$S_1^l(x) = \overline{X}^{-1}(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right], \quad \forall l \geq l^*.$$

Since S_1^l is player $1's$ stage game best response in stage l, (3.4) implies that,

$$S_1^l(x) \geq \overline{X}^{-1}(x), \quad \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right], \quad \forall l \geq l^*.$$

Suppose that in some equilibrium $\exists\ x \in [\underline{\underline{X}}, \overline{\overline{X}}]$ such that in stage l, $l \geq l^*$, $S_1^l(x) > \overline{X}^{-1}(x)$. Since player 1 cannot win, we must have,

$$S_{2a}^l(\underline{Y}(x)) \geq S_1^l(x).$$

Hence,

$$S_{2a}^l(\underline{Y}(x)) > \overline{X}^{-1}(x) = \underline{X}^{-1}(\underline{Y}(x)).$$

Therefore, when $\tilde{X}_2 = \underline{Y}(x)$, player $2A$ bids more than what he knows is the highest possible realization of $\tilde{V}$, i.e., $\underline{X}^{-1}(\underline{Y}(x))$, and he may win and pay a price greater than this. Hence he would be strictly better off in the stage game if he bid slightly less than $S_{2a}^l(\underline{Y}(x))$. S_{2a}^l cannot be his stage game best response. But we know that his best response is his stage game best response. Contradiction. ■

5 The Existence of an Equilibrium

The existence of an equilibrium can be shown by construction. The following is an equilibrium

$$\begin{aligned}
S_1^l(x) &= \overline{X}^{-1}(x), && \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right],\ l \geq 0,\\
S_{2b}^l(x) &\geq \underline{X}^{-1}(x), && \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right],\ l \geq 0,\\
S_{2a}^o(x) &= \underline{X}^{-1}(x), && \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right],\\
S_{2a}^l(x) &= S_{2b}^l(x), && \forall x \in \left[\underline{\underline{X}}, \overline{\overline{X}}\right],\ l \geq 1.
\end{aligned}$$

When these strategies are played, player 1 never wins. His payoff is zero. If he bids anything greater than the above strategy his expected payoff is strictly negative because if he wins, the price he pays is greater than or equal to the highest possible value of the object consistent with his signal. Similarly, players $2A$ and $2B's$ strategies are their best responses.

6 Concluding Remarks

When the same bidders encounter each other in a series of independent auctions, new phenomena may occur if there is incomplete information about one or more bidders' type. Bidders realize that their bid in any auction conveys information about their type. This is taken into account when computing optimal strategies. In this paper, a repeated, common-value, second-price auction model with two bidders was examined to capture some of these effects. A little incomplete information about the distribution of the true value of one of the players changes the game entirely, with severe consequences for the uninformed player 1. He will not win, except possibly in the last few stages and then only if player 2 is of the ordinary type. Even in these last few stages, the equilibrium is unfavorable to player 1 when compared to the symmetric equilibrium of the complete information game. Player 2, on the other hand, is in a good position. He wins more often and whenever he does, pays a lower price. It does not cost him anything in the stage game to maintain his reputation as an aggressive bidder. He plays his stage game best response in every stage. None of these results depend on the amount of initial uncertainty about player $2's$ type, or on the value of k. Two things are crucial — the common-value type set-up which gives rise to the winner's curse, and the fact that the auctions are second-price. Both make reputation building very profitable for player 2.

It is easy to show that the auctioneer's revenues in our model, are considerably lower than under the symmetric equilibrium of the symmetric game, because in the former case, the price paid will usually be player $1's$ bid, which is much lower. Also, the symmetric equilibrium of a symmetric, common-value, second-price auction with two risk-neutral bidders is unstable under the kind of departure from symmetry considered here. The symmetric equilibrium is not close to any of the equilibria in asymmetric stage games in which there is an arbitrarily small probability that bidder 2 may value the object more by an arbitrarily small amount.

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