UNCERTAINTY AND INSURANCE IN STRATEGIC MARKET GAMES

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Abstract

For perfectly competitive economies under uncertainty, there is a well-known equivalence between a formulation with contingent goods and one with state-specific securities followed by spot markets for goods. In this paper, I examine whether this equivalence carries over to a particular form of imperfect competition. Specifically, I look at three Shapley-Shubik strategic market games: one with contingent commodities, one with Arrow securities traded under imperfect competition and one with Arrow securities traded under perfect competition. First I compare the feasibility constraints of these three games. Then I compare their equilibrium sets. As in Peck and Shell (1989), the only common equilibria between the first and the second game are those which involve no transfer of income across states. However, if the securities markets are competitive, then the set of equilibria of the contingent commodities game and the securities game coincide.

1 Introduction

In the theory of general equilibrium, prices are non-strategic and determined by the invisible hand. While this may be an appropriate assumption for markets with many participants since each individual has little power to affect prices, it is less satisfactory when prices form between a small number of individuals. The strategic market games literature, pioneered by Shapley and Shubik, contains a variety of models with a particular mechanism of non-cooperative strategic price formation. These models have the attractive property that under fairly general assumptions their equilibria converge to a competitive equilibrium as the

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players are replicated to infinity. A natural question to ask is whether properties of the competitive model are preserved in these strategic market games. In this paper I look at one such property concerning the organization of markets under uncertainty.

It is a well known fact since the work of Arrow [1] and Debreu [2] in the fifties that under perfect competition and uncertainty, a pure exchange economy can be organized in two fundamentally different ways without disturbing the set of competitive equilibria. The first economy has individuals trading contingent commodities, or contracts to deliver one unit of a particular commodity in a particular state. The second economy has individuals trading Arrow securities first, and then trading actual commodities on spot markets after the state is revealed. An Arrow security is a contract to deliver a unit of money in a particular state. The set of competitive equilibria of these economies are identical, in terms of allocations.

The purpose of this paper is to examine whether this equivalence holds under imperfect competition, in the specific context of Shapley-Shubik strategic market games. I describe three games. Game I has only contingent commodities. Game II has Arrow securities traded on imperfectly competitive markets and game III has Arrow securities traded on perfectly competitive markets. In games II and III, goods are then traded on spot markets, after the uncertainty is revealed. In all three games, the goods markets are imperfectly competitive.

I first show that the three games have identical sets of budget-feasible allocations. This implies that any differences in the equilibrium sets are due to strategic considerations and not solely to feasibility constraints. I then compare the equilibrium sets. Game I and game II have different equilibrium sets. In fact the only common equilibria are those which exhibit no transfer of income between states. Peck and Shell [3] contains a similar result with a different proof. I show that game I and game III have identical equilibrium sets.

To get some intuition for these results, observe the following. The equilibria of a simple market game without uncertainty are in general not the same as the competitive equilibria of the corresponding economy. Now take two securities games which have the same rules of price formation on the goods markets but of which one has Shapley-Shubik rules of price formation on the asset markets (game II) and the other has competitive asset markets (game III). The equilibria of these two games may not be the same, just like the equilibria of the basic game are different from the corresponding competitive equilibria. This is in fact what I show. I compare each of the securities games to the contingent commodities game and I show that in one case the equilibrium sets are almost disjoint and in the other case they are the same, which establishes the result.

These results put into perspective the importance of the extent of strategic power that players have. Individuals have a different kind of strategic power in games I and III than in game II. In games I and III, individuals can affect the relative prices of all goods and only that: the perfectly competitive securities markets of game III introduce exogenous multiplicative factors for each state. Because in game II there is an imperfectly competitive securities market, players not only affect the relative prices of all commodities but they also affect the relative prices of sets of commodities, namely the commodities sold in any particular state. Therefore it seems that it is not the way markets are organized which alters the set of equilibria, as Peck and Shell claim, but the nature and amount of strategic power.

In section 2, I present the models. In section 3, I show that the three games have identical budget-feasible sets of allocations. In section 4, I compare the contingent commodity game to the Arrow securities game with imperfect competition. In section 5, I compare the contingent commodity game to the perfectly competitive Arrow securities game. Section 6 is the conclusion.

2 The Models

Consider the following economy. There are r states of the world indexed by s, with t referring to a particular state, there are m goods indexed by k and finally, there are n individuals indexed by i or j.

Each individual is an expected-utility maximizer, with respect to a strictly positive prior π^i, and receives an endowment of commodities in all states, with $e^i_{k,s}$ denoting individual i's endowment of good k in state s.

Individual i's consumption of good k in state s will be denoted by $\xi^i_{k,s}$, with $\xi^i_s = (\xi^i_{k,s})^m_{k=1}$. Hence his utility is given by

$$\mathcal{U}^i(\xi^i) = \sum_s \pi^i(s) u^i(\xi^i_s).$$

I assume throughout the paper that there are at least two individuals ($n \geq 2$) and two states ($r \geq 2$). I will use three additional assumptions in places and will specify when I use them.

(A1) u^i is differentiable on the strictly positive orthant of $\mathcal{R}^m$, $\forall i$.
(A2) u^i is non-decreasing in each variable, $\forall i$.
(A3) u^i is strictly increasing, $\forall i$.

Definition 1. *An allocation is said to be achievable in a game if there is a strategy profile in that game which results in the given allocation.*

Now I am ready to describe the games.

2.1 The Contingent Commodities Game: G^I

In this game, individuals trade contingent commodities and receive contingent consumption vectors. Then the state is revealed and the goods corresponding to that state are actually delivered and consumed. The rules of price formation and trade are those put forth by Shapley and Shubik, with one difference. Shapley and Shubik use a commodity-money, cash-in-advance model. For reasons which will be discussed later, the model I use here has fiat money in total zero quantity hence with budget constraints and bankruptcy rules.

Formally, a strategy profile for individual i is a collection of $r \cdot m$ pairs of bids and offers $(b^i_{k,s}, q^i_{k,s})$ for each good in each state, i.e. $s^i = ((b^i_{k,s}, q^i_{k,s})^m_{k=1})^r_{s=1}$. The price of good k in state s is given by

$$p_{k,s} = \begin{cases} \frac{\sum_i b^i_{k,s}}{\sum_i q^i_{k,s}} & \text{if } \sum_i q^i_{k,s} > 0, \\ 0 & \text{else.} \end{cases}$$

If $\sum_i q^i_{k,s} = 0$ then all bids for good (k,s) are confiscated and if $\sum_i b^i_{k,s} = 0$ then all offers of good (k,s) are confiscated as well. This is a standard assumption in the literature. A strategy profile must satisfy the following constraints:

$$\begin{aligned} b^i_{k,s} &\geq 0, \quad \forall i,k,s, \\ e^i_{k,s} \geq q^i_{k,s} &\geq 0, \quad \forall i,k,s. \end{aligned}$$

In addition to that, each individual has a global budget constraint which must hold in order for trade to generate positive (contingent) consumption vectors:

$$\sum_s \sum_k (b^i_{k,s} - p_{k,s} q^i_{k,s}) \leq 0 \quad \forall i. \qquad (*^i)$$

Note that this takes into account the fact that bids for a good have to be paid even if the offers of that good and hence the price are zero.

Strategies which satisfy this constraint are called budget-feasible and generate consumption as follows:

$$\xi^i_{k,s} = \begin{cases} e^i_{k,s} - q^i_{k,s} + \frac{b^i_{k,s}}{p_{k,s}} & \text{if } (*^i) \text{ holds and } p_{k,s} > 0, \\ e^i_{k,s} - q^i_{k,s} & \text{if } (*^i) \text{ holds and } p_{k,s} = 0, \\ 0 & \text{else.} \end{cases}$$

The fact that all goods are confiscated if the global budget constraint is not satisfied is not essential. Any bankruptcy rule which makes it impossible to have bankrupt individuals at equilibrium will do.

The following notation will be useful later. Let $Q^i_{k,s}$ denote individual i's net trade of good k in state s, i.e.

$$Q^i_{k,s} = \xi^i_{k,s} - e^i_{k,s}$$

Finally, let $A(G^I)$ denote the set of payoff vectors which can be collectively achieved in G^I and $\Xi(G^I)$ the set of payoff vectors which can be collectively achieved in G^I with budget-feasible strategy profiles. The equilibrium concept used for this game is standard pure strategy Nash equilibrium.

2.2 Imperfect Competition in the Arrow Securities Game: G^{II}

In this game, individuals first trade inside securities, and then, after the state of the world is revealed, they trade goods in that state on spot markets. In both stages, the rules of price formation are those of Shapley and Shubik.

The bids for securities are made in terms of a fiat money of which each individual has a total zero quantity. In other words, purchases of securities must be financed by the sales of other securities.

In the second stage, the state t is revealed and each state-t security gives one unit of account in state t while other securities give zero. Individuals then bid for state-t commodities in terms of their state-t money. The sum of the purchases made in state t in the second stage and payments for t-securities sold in the first stage must be financed by the combined income of sales of goods in state t in the second stage and income from t-securities purchased in the first stage.

A strategy for individual i consists of pairs of bids and offers (b_s^i, q_s^i) for every state-specific security and pairs of bids and offers for every good in every state. Formally,

$$s^i = ((b_s^i, q_s^i); (\overline{b}_{k,s}^i, \overline{q}_{k,s}^i)_{k=1}^m)_{s=1}^r.$$

The price of each security is given by

$$p_s = \begin{cases} \frac{\sum_i b_s^i}{\sum_i q_s^i} & \text{if } \sum_i q_s^i > 0, \\ 0 & \text{else,} \end{cases}$$

and the price of each good is similarly given by

$$\overline{p}_{k,s} = \begin{cases} \frac{\sum_i \overline{b}_{k,s}^i}{\sum_i \overline{q}_{k,s}^i} & \text{if } \sum_i \overline{q}_{k,s}^i > 0, \\ 0 & \text{else.} \end{cases}$$

Note that here again, on the securities markets as well as on the goods markets, if the total offer of one item is zero, then the bids for this item are confiscated and vice versa.

A strategy profile must satisfy the following constraints:

$$\begin{array}{rcl} b_s^i & \geq & 0 \quad \forall i, s, \\ q_s^i & \geq & 0 \quad \forall i, s, \\ \overline{b}_{k,s}^i & \geq & 0 \quad \forall i, k, s, \\ e_{k,s}^i \geq \overline{q}_{k,s}^i & \geq & 0 \quad \forall i, k, s. \end{array}$$

In addition, there are a set of budget constraints which must hold in order for the strategy to be budget-feasible and for trade to generate consumption:

$$\sum_k (\overline{b}_{k,s}^i - \overline{p}_{k,s}\overline{q}_{k,s}^i) \leq \frac{b_s^i}{p_s} - q_s^i \qquad \text{if } p_s > 0, \tag{$*^{is}$}$$

$$\sum_k (\overline{b}_{k,s}^i - \overline{p}_{k,s}\overline{q}_{k,s}^i) \leq -q_s^i \qquad \text{if } p_s = 0, \tag{$*_0^{is}$}$$

$$\sum_s (b_s^i - q_s^i p_s) \leq 0. \tag{$**^i$}$$

For a budget-feasible strategy, consumption is then determined as follows:

$$\overline{\xi^i_{k,s}} = \begin{cases} e^i_{k,s} - \overline{q}^i_{k,s} + \frac{\overline{b}^i_{k,s}}{\overline{p}_{k,s}} & \text{if } (*^{is})/(*^{is}_0) \text{ and } (**^i) \text{ hold and } p_{k,s} > 0, \\ e^i_{k,s} - \overline{q}^i_{k,s} & \text{if } (*^{is})/(*^{is}_0) \text{ and } (**^i) \text{ hold and } p_{k,s} = 0, \\ 0 & \text{else.} \end{cases}$$

Constraint $(**^i)$ represents the notion that individuals have zero money and hence must finance their bids on securities by sales of securities. Constraint $(*^{is})$ captures the notion that the net spending on goods in state s must be financed by the net income from state s securities or conversely, that the net revenue from the sales of goods in state s must be sufficient to finance the net debt from sales of state s securities. Constraint $(*^{is}_0)$ is for the case where the price of the s-security is zero and hence the offers are confiscated if they are positive.

Once again, it is useful to have notation for the net trades. Let Q^i_s denote individual i's net purchase of security s and let $\overline{Q}^i_{k,s}$ denote individual i's net purchase of good k in state s. Formally, let

$$Q^i_s = \begin{cases} \frac{b^i_s}{p_s} - q^i_s & \text{if } p_s > 0, \\ -q^i_s & \text{else,} \end{cases}$$

and

$$\overline{Q}^i_{k,s} = \overline{\xi^i_{k,s}} - e^i_{k,s}.$$

Finally, let $A(G^{II})$ denote the set of payoff vectors which can be collectively achieved in G^{II} and $\Xi(G^{II})$ the set of payoff vectors which can be collectively achieved in G^{II} with budget-feasible strategy profiles. Here again, the equilibrium concept is pure strategy Nash Equilibrium.

2.3 Perfect Competition in The Arrow Securities Game: $G^{III}_{p^*}$

This game is very similar to the previous one, G^{II}. The only difference lies in the prices of the securities. In this game, these prices are fixed exogenously. Unless explicitly specified, these prices are assumed to be strictly positive. The importance of this assumption should be clear. If the price of a security is zero, the corresponding state is in effect isolated and the corresponding economy is altered.

I use the same symbols to denote quantities in G^{II} and in $G^{III}_{p^*}$. In what follows, it should be clear from the context whether I am referring to G^{II} or $G^{III}_{p^*}$.

A strategy for individual i is still of the form

$$s^i = ((b^i_s, q^i_s); (\overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})^r_{s=1}.$$

The prices for securities are now exogenously given by

$$p_s = p_s^* > 0 \quad \forall s,$$

and the price of each good is given in the same way as before by

$$\overline{p}_{k,s} = \begin{cases} \frac{\sum_i \overline{b}^i_{k,s}}{\sum_i \overline{q}^i_{k,s}} & \text{if } \sum_i \overline{q}^i_{k,s} > 0, \\ 0 & \text{else.} \end{cases}$$

The feasibility and budget constraints are the same here as in G^{II} except for the fact that the security prices are now exogenously given by p^*. Let $A(G^{III}_{p^*})$ denote the set of payoff vectors which can be collectively achieved in $G^{III}_{p^*}$ and $\Xi(G^{III}_{p^*})$ the set of payoff vectors which can be collectively achieved in $G^{III}_{p^*}$ with budget-feasible strategy profiles.

For this game, the equilibrium concept should be mixed. Indeed, given the securities prices $p^*_s, \forall s$ the strategies still have to form a Nash Equilibrium and in addition to that, the securities markets have to clear at the prices $p^*_s, \forall s$. However, given the bankruptcy rule and the hypotheses on the utility functions, equilibrium strategies must be budget feasible. In the next result, I show that budget feasible strategies must clear all markets, for any securities prices. Hence the relevant equilibrium concept is still Nash equilibrium.

Proposition 1. *Budget-feasible strategy profiles in $G^{III}_{p^*}$ clear goods markets and securities markets. Formally, if for all i:*

$$\begin{cases} \sum_k (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) \leq \frac{b^i_s}{p^*_s} - q^i_s, & \forall s, \\ \sum_s (b^i_s - p^*_s q^i_s) \leq 0 \end{cases}$$

then

$$\sum_i \overline{b}^i_{k,s} = \overline{p}_{k,s} \sum_i \overline{q}^i_{k,s}, \forall k, s$$

and

$$\sum_i b^i_s = p^*_s \sum_i q^i_s, \forall s.$$

Proof: Take a budget-feasible strategy profile in $G^{III}_{p^*}$, let $D^i_s = b^i_s - p^*_s q^i_s$ and observe that $\sum_s D^i_s \leq 0$. The rules of price formation tell us that $\sum_i (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) \geq 0$. Budget-feasibility also tells us that $\sum_k (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) \leq \frac{D^i_s}{p^*_s}$. Hence, because $p^*_s > 0$, we can multiply the previous two inequalities by p^*_s. Then by summing over goods and states for the first one and over states and individuals for the second one, we see that

$$\begin{aligned} 0 \leq \sum_s p^*_s \sum_k \sum_i (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) &\leq \sum_s \sum_i D^i_s \leq 0 \\ \Rightarrow \sum_s p^*_s \sum_k \sum_i (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) = 0 &= \sum_s \sum_i D^i_s \\ \Rightarrow \sum_i (\overline{b}^i_{k,s} - \overline{p}_{k,s}\overline{q}^i_{k,s}) = 0 &\quad, \forall k, s \end{aligned}$$

and the goods markets must clear. Now, again by budget-feasibility, observe that

$$0 = p_s^* \sum_i \sum_k (\overline{b}_{k,s}^i - \overline{p}_{k,s}\overline{q}_{k,s}^i) \leq \sum_i (b_s^i - p_s^* q_s^i)$$

and

$$\sum_s \sum_i (b_s^i - p_s^* q_s^i) = 0,$$

hence

$$\sum_i (b_s^i - p_s^* q_s^i) = 0, \forall s$$

and the securities markets clear as well. ■

Similar results holds for G^I and G^{II}, with similar proofs.

3 Payoff Equivalence

In this section, I show that the set of payoffs which can be achieved with budget-feasible strategy profiles is the same in all three games. To do this, I take a budget-feasible strategy profile in G^I and define a strategy profile in G^{II} or $G_{p^*}^{III}$ which I show to be budget-feasible and which generates the same payoff vectors. Conversely from G^{II} or $G_{p^*}^{III}$ to G^I.

It is important to understand that the budget-feasible strategy profiles which lead to the same payoff vectors are equivalent collectively. In other words, one needs the whole strategy profile in one game in order to define any component of the corresponding strategy profile in the other game.

Also important is the obvious redundancy, already observed by Shapley and Shubik, built into the model. Given a strategy profile, a price is computed for each good. Neither the price nor the final allocations change if individuals modify their bids and offers by quantities which are proportional to the price of that good. In the model with Arrow securities traded on imperfectly competitive markets, the same redundancies appear in the markets for goods and additional redundancies of a similar form appear in the markets for securities.

In what follows, I show that the same budget-feasible payoff vectors can be attained in all three games. Note that I do not claim to identify the set of strategies leading to the same final allocation.

Proposition 2. *The set of payoffs resulting from budget-feasible strategy profiles is the same in all three games.*

Proof: Given $s = (s^i)_{i \in I}$ a budget-feasible strategy profile in G^I, let $\xi = (\xi^i)_{i \in I}$ denote the corresponding vector of final allocations. I define $t = (t^i)_{i \in I}$ in G^{II} or in $G_{p^*}^{III}$ for $p_s^* = 1$ as follows:

$$\begin{array}{rcl} b^i_s & = & \sum_k b^i_{k,s} \quad \forall i, s, \\ q^i_s & = & \sum_k p_{k,s} q^i_{k,s} \quad \forall i, s, \\ \overline{b}^i_{k,s} & = & b^i_{k,s} \quad \forall i, k, s, \\ \overline{q}^i_{k,s} & = & q^i_{k,s} \quad \forall i, k, s. \end{array}$$

I then show that t is budget-feasible and generates ξ.

To go in the other direction, take $t = (t^i)_{i \in I}$, a budget-feasible strategy profile in G^{II} or $G^{III}_{p^*}$ and define $s = (s^i)_{i \in I}$ in G^I as follows, with p_s denoting the positive price of the securities in both games, in the interest of space:

$$\begin{array}{rcl} b^i_{k,s} & = & p_s \overline{b}^i_{k,s} \\ q^i_{k,s} & = & \overline{q}^i_{k,s}. \end{array}$$

■

It is now clear that equilibrium payoffs of any of these games are achievable in the other two. Any differences in equilibrium sets would therefore be due at least in part to strategic considerations.

Note that it is important that p^* be strictly positive. Indeed, having $p^*_s = 0$ for some s is tantamount to "cutting" the game in parts – one part for each s such that $p^*_s = 0$ and one part with the rest – and playing the different parts separately. Now each part has the same set of budget-feasible payoffs as the corresponding contingent commodities game. Indeed, the part corresponding to the positive security prices satisfy the hypotheses and the parts that have just one state are simply identical to their contingent commodities counterpart. It is clear that there is no need for the "broken-up" contingent commodities game to be equivalent to its original form and that in general the two have different sets of budget-feasible payoffs and certainly different sets of equilibria.

Proposition 2 sets the stage by showing equivalent budget-feasibility conditions for the three games. Note that this does not entirely disentangle feasibility considerations from strategic considerations. Take a non-equilibrium budget-feasible strategy profile in one of the games and consider the strategy profile representing a profitable deviation for one of the players. There is no reason for this to be budget-feasible for all players. Indeed, the deviation of a player may cause other players to violate some of their budget constraints. Hence this altered strategy profile may lead to a payoff which is not achievable in the other games. It is thus possible that an equilibrium payoff of one game is not an equilibrium payoff of another game, because of different feasibility constraints.

Feasibility considerations cannot be completely disentangled from strategic considerations because it is not the case that the sets of achievable payoffs are the same in all three games. Indeed, in the securities games, if an individual violates a state budget constraint without violating his global constraint, his goods are confiscated only in that state. This type of payoff cannot be replicated in the contingent commodity game. Also, actions on the securities markets play out differently in G^{II} and $G^{III}_{p^*}$.

The following result shows nonetheless that the extent to which equilibria of $G^{III}_{p^*}$ are not equilibria of G^{II} or G^{I} is due to strategic considerations alone.

Proposition 3. *The sets of payoffs which are achievable in games I and II are each included in the set of payoffs which are achievable in game III. Formally, with $A(G^I)$, $A(G^{II})$ and $A(G^{III}_{p^*})$ denoting the set of achievable payoffs for games I, II and III respectively,*

$$\begin{array}{lrcl} 1. & A(G^I) & \subseteq & A(G^{III}_{p^*}) \\ 2. & A(G^{II}) & \subseteq & A(G^{III}_{p^*}) \end{array}$$

Proof:
1. Take $s = ((b^i_{k,s}, q^i_{k,s})^m_{k=1})^r_{s=1}$, a strategy profile in G^I leading to prices $p_{k,s}$. Now, set $p^* = 1$ and let $t = ((\sum_k b^i_{k,s}, \sum_k p_{k,s}q^i_{k,s}); (b^i_{k,s}, q^i_{k,s})^m_{k=1})^r_{s=1}$ denote a strategy profile in $G^{III}_{p^*}$. It is easy to verify that either both of these strategy profiles are budget-feasible or neither is, and that they lead to the same final allocations.
2. Take $t = ((b^i_s, q^i_s); (\overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})^r_{s=1}$, a strategy profile in G^{II} leading to security prices p_s. Set $p^*_s = 1, \forall s$, and let τ denote a strategy profile in $G^{III}_{p^*}$ given by the following state-by-state components:

for $p_s > 0$: $((b^i_s, p_s q^i_s); (p_s \overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})$
for $p_s = 0$,

if i violates his global budget constraint in G^{II}: $((b^i_s + q^i_s, q^i_s); (\overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})$,
if i meets his global budget constraint in G^{II}: $((0, q^i_s); (\overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})$.

It is easy, even if tedious, to verify that t and τ have the same budget constraints and lead to the same vector of final allocations. ■

Now take t, an equilibrium of $G^{III}_{p^*}$ leading to a payoff ξ. Because t is an equilibrium, it must be budget-feasible in $G^{III}_{p^*}$. Hence by proposition 2, ξ can be reached in G^I and G^{II} with budget-feasible strategy profiles. If none of these strategy profiles are equilibria in G^I or G^{II}, then for each of them an individual must have a profitable deviation. An allocation resulting from one of these deviations is achievable in $G^{III}_{p^*}$ by the above proposition but cannot result from a deviation from t by the individual who prefers this allocation to the one resulting from t, by assumption that t is an equilibrium. Hence if ξ is not an equilibrium payoff of G^I or G^{II}, it must be because of strategic considerations.

The next two sections clarify the strategic considerations which affect the equivalence (or not) of the equilibrium sets of these games.

4 Equilibria of G^I and G^{II}: A Near Non-Equivalence Result

In this section, I compare interior equilibria of G^I and G^{II}. An interior equilibrium is one where all individuals consume in the interior of their consumption set and all markets are open; this requires all prices to be strictly positive. I first exhibit the necessary conditions for interior equilibria of both games. Then I assume that a particular allocation is an interior equilibrium allocation of both games, resulting from strategies with equal offers of both goods, and I examine the implications of this fact.

The crucial implication is that equilibria of both games, resulting from such strategies, can only be the same at an optimal allocation of income. This in turn is not compatible with perceived market power in the securities market. Finally, I show that if individuals are not perceiving market power at equilibrium, they must not be trading any securities.

This result contrasts with the competitive analog where optimal allocation of income is compatible with insurance, or the trading of securities.

In this section, I assume (A1) and (A3), i.e. that the u^i are differentiable and strictly increasing.

4.1 The maximization problem in G^I

Let $\mathcal{P}_i^I$ denote the maximization problem of individual i in G^I. Individual i maximizes his expected utility of final allocations subject to a global budget constraint and to the rules of price formation and trade. Hence, $\mathcal{P}_i^I$ is given by

$$\begin{array}{ll} \text{Max} & \mathcal{U}^i(\xi^i) \\ \text{s.t.} & \sum_s \sum_k p_{k,s} Q^i_{k,s} \leq 0 \quad (\lambda^i) \\ & e^i_{k,s} \geq q^i_{k,s} \geq 0 \\ & b^i_{k,s} \geq 0 \end{array}$$

where

$$p_{k,s} = \frac{\sum_i b^i_{k,s}}{\sum_i q^i_{k,s}}, \text{ which is assumed to be } > 0,$$

$$Q^i_{k,s} = \frac{b^i_{k,s}}{p_{k,s}} - q^i_{k,s}.$$

I assume positive prices because I only look at interior equilibria.

At an interior equilibrium, the Lagrangian can be written as

$$\mathcal{L}_i^I = \mathcal{U}^i(\xi^i) - \lambda^i(\sum_s \sum_k p_{k,s} Q^i_{k,s})$$

and must satisfy

$$\frac{d\mathcal{L}_i^I}{db^i_{k,s}} = \frac{d\mathcal{L}_i^I}{dq^i_{k,s}} = 0, \forall i, k, s.$$

It is easy – even if tedious – to show that an interior equilibrium of G^I must satisfy the conditions

$$\mathcal{U}^i_{k,s} = \lambda^i \frac{q^{-i}_{k,s}}{b^{-i}_{k,s}} p^2_{k,s}, \qquad \forall i, k, s,$$

where $\mathcal{U}^i_{k,s}$ denotes $\frac{d\mathcal{U}^i}{dQ^i_{k,s}}$. Since u^i is assumed to be strictly increasing, we must have $\lambda^i > 0$ and hence

$$\sum_s \sum_k p_{k,s} Q^i_{k,s} = 0.$$

4.2 The maximization problem in G^{II}

In G^{II}, individuals face a slightly different set of constraints. Instead of a global budget constraint, they face a budget constraint in each state but they jointly determine these state-by-state budget constraints in the first stage of the game. In addition, each individual faces a budget constraint on securities. Hence $\mathcal{P}^{II}_i$ is given by

$$\begin{array}{ll}
\text{Max} & \mathcal{U}^i(\xi^i) \\
\text{s.t.} & \sum_k \overline{p}_{k,s} \overline{Q}^i_{k,s} \leq Q^i_s \quad (\lambda^i_s) \\
& \sum_s p_s Q^i_s \leq 0 \quad (\mu^i) \\
& e^i_{k,s} \geq \overline{q}^i_{k,s} \geq 0 \\
& \overline{b}^i_{k,s} \geq 0 \\
& b^i_s \geq 0 \\
& q^i_s \geq 0
\end{array}$$

where $\overline{Q}^i_{k,s}$ and $\overline{p}_{k,s}$ are analogous to $Q^i_{k,s}$ and $p_{k,s}$ in $\mathcal{P}^I_i$, and

$$\begin{aligned}
p_s &= \frac{\sum_i b^i_s}{\sum_i q^i_s}, \\
Q^i_s &= \frac{b^i_s}{p_s} - q^i_s.
\end{aligned}$$

The Lagrangian can be written as

$$\mathcal{L}^{II}_i = \mathcal{U}^i(\xi^i) - \sum_s \lambda^i_s (\sum_k \overline{p}_{k,s} \overline{Q}^i_{k,s} - Q^i_s) - \mu^i (\sum_s p_s Q^i_s).$$

Note here that at an interior equilibrium all markets are open, including all securities markets. Such an equilibrium must satisfy

$$\frac{d\mathcal{L}^{II}_i}{d\overline{b}^i_{k,s}} = \frac{d\mathcal{L}^{II}_i}{d\overline{q}^i_{k,s}} = 0 \quad \forall i, k, s,$$

and

$$\frac{d\mathcal{L}^{II}_i}{db^i_s} = \frac{d\mathcal{L}^{II}_i}{dq^i_s} = 0 \quad \forall i, s.$$

Again, because u^i is strictly increasing, we have $\lambda_s^i, \mu^i > 0$ and hence the necessary conditions for an interior equilibrium are

$$\begin{aligned}
\overline{\mathcal{U}}_{k,s}^i &= \lambda_s^i \frac{\overline{q}_{k,s}^{-i}}{\overline{b}_{k,s}^{-i}} \overline{p}_{k,s}^2, \\
\lambda_s^i &= \mu^i \frac{q_s^{-i}}{b_s^{-i}} p_s^2, \\
\sum_k \overline{p}_{k,s} Q_{k,s}^i &= Q_s^i, \text{ and} \\
\sum_s p_s Q_s^i &= 0,
\end{aligned}$$

where $\overline{\mathcal{U}}_{k,s}^i$ denotes $\frac{d\mathcal{U}^i}{d\overline{Q}_{k,s}^i}$

4.3 The near non-equivalence result

In this section, I show that strategy profiles which constitute interior equilibria of both games and involve identical offers of each good in each state by each player cannot involve any transfer of income across states.

Definition 2. *A strategy profile in G^I and one in G^{II} are said to be offer-consistent if every individual offers the same amount of each good in every state in both games. That is is*

$$q_{k,s}^i = \overline{q}_{k,s}^i, \forall i, k, s.$$

The purpose of this restriction is to avoid the confusion that could result from the redundancies mentioned above. For given final allocations and given prices, the bids and offers of each individual can still vary by quantities which are proportional to the price, leaving both price and final allocations unchanged. Pinning down the offers eliminates the redundancy, even if not in an innocuous manner. I am now ready to state and prove the main result of this section.

Proposition 4. *Assume (A1) and (A3). If two offer-consistent strategy profiles in G^I and G^{II} constitute interior equilibria leading to the same final allocations, then they also exhibit no trade on the securities markets. Formally,*

$$\begin{aligned}
\textit{If} \quad (s^i)_{i\in I} \quad &\textit{and} \quad (\overline{s}^i)_{i\in I} \textit{ are interior equilibria of } G^I \textit{ and } G^{II}, \\
Q_{k,s}^i \quad &= \quad \overline{Q}_{k,s}^i \quad \forall i, k, s \textit{ and} \\
q_{k,s}^i \quad &= \quad \overline{q}_{k,s}^i \quad \forall i, k, s, \\
\textit{Then} \quad Q_s^i \quad &= \quad 0, \quad \forall i, s.
\end{aligned}$$

Proof:
Recall that in G^I the first-order conditions are given by

$$\frac{d\mathcal{U}^i}{dQ_{k,s}^i} = \lambda^i \frac{q_{k,s}^{-i}}{b_{k,s}^{-i}} p_{k,s}^2,$$

and in G^{II} they are given by

$$\frac{d\mathcal{U}^i}{d\overline{Q}^i_{k,s}} = \lambda^i_s \frac{\overline{q}^{-i}_{k,s}}{\overline{b}^{-i}_{k,s}} \overline{p}^2_{k,s},$$

and

$$\lambda^i_s = \mu^i \frac{q^{-i}_s}{b^{-i}_s} p^2_s.$$

1. $Q^i_{k,s} = \overline{Q}^i_{k,s}$ and both are interior equilibria imply that

$$\lambda^i \frac{q^{-i}_{k,s}}{b^{-i}_{k,s}} p^2_{k,s} = \lambda^i_s \frac{\overline{q}^{-i}_{k,s}}{\overline{b}^{-i}_{k,s}} \overline{p}^2_{k,s} \tag{1}$$

2. $q^i_{k,s} = \overline{q}^i_{k,s}$ implies that (1) becomes

$$\lambda^i \frac{p^2_{k,s}}{b^{-i}_{k,s}} = \lambda^i_s \frac{\overline{p}^2_{k,s}}{\overline{b}^{-i}_{k,s}} \tag{2}$$

3. $q^i_{k,s} = \overline{q}^i_{k,s}$ and $Q^i_{k,s} = \overline{Q}^i_{k,s}$ imply

$$\frac{b^i_{k,s}}{p_{k,s}} = \frac{\overline{b}^i_{k,s}}{\overline{p}_{k,s}}, \forall i \Rightarrow \frac{b^{-i}_{k,s}}{p_{k,s}} = \frac{\overline{b}^{-i}_{k,s}}{\overline{p}_{k,s}}, \forall i \tag{3}$$

4. $(2) + (3) \Rightarrow (4)$

$$\lambda^i p_{k,s} = \lambda^i_s \overline{p}_{k,s} \tag{4}$$

$$\Rightarrow \frac{p_{k,s}}{\overline{p}_{k,s}} = \frac{\lambda^i_s}{\lambda^i}$$

which depends only on s and hance we can define $c_s = \frac{\lambda^i_s}{\lambda^i}$. This implies that

$$\frac{\lambda^i_s}{\lambda^i_t} = \frac{c_s}{c_t},$$

the right hand side of which is independent of i, and hence income must be optimally allocated across states. Indeed, λ^i_s is individual i's marginal utility of income in state s and the last equality tells us that the ratio of state-s and state-t marginal utilities of income must be constant across individuals.

Incidentally, (4) also implies that given a state s, the relative prices of all goods in that state are the same in both games.

5. Let $A^i = \frac{\lambda^i}{\mu^i}$ and $k_s = \frac{c_s}{p_s}$. Then

$$\lambda^i_s = \mu^i \frac{q^{-i}_s}{b^{-i}_s} p^2_s = \lambda^i c_s \tag{5}$$

$$\Rightarrow \frac{b_s^{-i}}{q_s^{-i}} = (\frac{\mu^i}{\lambda^i})(\frac{p_s}{c_s})p_s = \frac{1}{A^i}k_s p_s \tag{6}$$

Note that the only way for $\frac{b_s^{-i}}{q_s^{-i}}$ to be constant across individuals is for $\frac{b_s^i}{q_s^i}$ to be constant across individuals as well. But in that case we must have $\frac{b_s^i}{q_s^i} = p_s \Leftrightarrow Q_s^i = 0$.

6. Note that

$$Q_s^i \geq 0 \quad \Leftrightarrow \quad \frac{b_s^i}{q_s^i} \geq p_s \quad \Leftrightarrow \quad \frac{b_s^{-i}}{q_s^{-i}} \leq p_s,$$

and hence

$$Q_s^i \geq 0 \quad \Leftrightarrow \quad k_s \leq A^i.$$

Now order the A^i's in decreasing order: $A^{(1)} \geq ... \geq A^{(n)}$, and observe that we must have $k_s \leq A^{(1)}, \forall s$.

Indeed, if $k_s > A^{(1)}$, for some s, then $Q_s^i < 0, \forall i$ and this is impossible because $\sum_i Q_s^i = 0$. Hence $k_s \leq A^{(1)}, \forall s$.

7. Now observe that

$$k_s \leq A^{(1)}, \forall s \Rightarrow Q_s^i \geq 0, \forall s, \text{ for } i \text{ such that } A^i = A^{(1)}.$$

Because all markets are open, $p_s > 0, \forall s$ and hence we also have

$$\sum_s p_s Q_s^i \leq 0 \Rightarrow Q_s^i = 0, \forall s.$$

Hence $k_s = A^{(1)}, \forall s$.

8.

$$k_s = A^{(1)} \geq A^{(2)} \geq ... \geq A^{(n)} \quad \forall s.$$

By the same reasoning as above, if $A^{(k)} \leq A^{(1)} = k_s \quad \forall k$ then $Q_s^j \leq 0, \forall j \neq i$ but $\sum_j Q_s^j = 0$ and hence $Q_s^j = 0, \forall j$. Therefore $A^i = A, \quad k_s = k$, and $A = k$ which confirms that $Q_s^i = 0$. Indeed, (6) then implies that $\frac{b_s^{-i}}{q_s^{-i}} = p_s \quad \forall s$, which in turn implies that $\frac{b_s^i}{q_s^i} = p_s \quad \forall s$.

Hence offer-consistent interior equilibria of the two games can only result in the same allocation at an optimal allocation of income and this is incompatible with any individual using his market power in the securities markets. In other words the price of each security has to be the same with and without the contribution of each individual. For this to be true, there can be no trading of securities.

9. Finally and for completeness, observe that (5) implies that in fact

$$\lambda_s^i = \mu^i p_s = \lambda^i c_s \quad \forall i, s.$$

Hence the marginal utility of income in state s is proportional to the price of security s, where the proportionality factor is the marginal utility of money to that individual.

This completes the proof. ■

It is worth noting that nowhere in the proof did I use the fact that there was more than one good in each state. It is really the fact that the value of money is determined in a non-competitive way that drives the result. Indeed, if there is only one good in each state, the proof goes through and I get the somewhat surprising result that choosing how much money to allocate to each good, knowing that the price of the good will be affected by this decision, is not the same as choosing how much money to allocate to each good knowing that both the value of the money and the price of the good will be affected by the decision.

On the other hand, the proof relies heavily on the assumption that there are at least two states. To see this, simply observe that if there is only one state, G^I and G^{II} are identical games. Indeed, there can be no trade on the security market in G^{II} and hence all the trade has to take place between the goods of the single state, in both games.

Also note that the main issue in this proof seems to be the excess of strategic power in G^{II}. Indeed, for the equilibria to be the same, individuals have to seemingly have the same amount of strategic power in both games. In other words, it has to be that no individual is using his ability to affect prices in the markets for securities. In the next section, $G^{III}_{p^*}$ has the same amount of strategic power as G^I in the sense that the normalization between states is exogenous so that in both cases each individual can affect the relative prices of all goods and nothing else. In that case the equilibrium sets are the same.

The same way that the equilibria of the basic game are not the same as the competitive equilibria of the corresponding economy, the equilibria of the Arrow securities game with perfect competition in the first stage are not the same as the equilibria of the Arrow securities game with an added element of imperfect competition at the securities level.

5 Equilibria of G^I and $G^{III}_{p^*}$: An Equivalence Result

In this section, I show that the sets of equilibrium allocations of these two games are identical. To do this, it is useful to have clearly in mind the way that I construct payoff-equivalent strategy profiles.

For $s = (s^i)_{i \in I} = (((q^i_{k,s}, b^i_{k,s})^m_{k=1})^r_{s=1})^n_{i=1}$ in G^I, fix $p^*_s > 0, \forall s$, and define $T(s)$

in $G^{III}_{p^*}$ as follows:

$$\begin{aligned}\overline{q}^i_{k,s} &= q^i_{k,s}, \forall i,k,s \\ \overline{b}^i_{k,s} &= \frac{b^i_{k,s}}{p^*_s}, \forall i,k,s \\ b^i_s &= \sum_k b^i_{k,s}, \forall i,s \\ q^i_s &= \frac{1}{p^*_s}\sum_k p_{k,s} q^i_{k,s}, \forall i,s.\end{aligned}$$

Conversely, for $t = (t^i)_{i\in I} = (((b^i_s, q^i_s); (\overline{b}^i_{k,s}, \overline{q}^i_{k,s})^m_{k=1})^r_{s=1})^n_{i=1}$ in $G^{III}_{p^*}$, define $S(t)$ as

$$\begin{aligned}q^i_{k,s} &= \overline{q}^i_{k,s}, \forall i,k,s \\ b^i_{k,s} &= p^*_s \overline{b}^i_{k,s}, \forall i,k,s.\end{aligned}$$

Now let $E(G^I)$ and $E(G^{III}_{p^*})$ denote the sets of equilibrium allocations of G^I and $G^{III}_{p^*}$ respectively. The next two propositions establish reciprocal inclusions of these two sets and hence give the desired result. The proof is done by contradiction. Take two payoff-equivalent strategy profiles of the above form and assume the second one is an equilibrium but not the first. Assume a budget-feasible deviation for one player. Then consider the second strategy profile and construct a deviation for the same player which is payoff equivalent to the original deviation. This shows that if one player has a budget-feasible profitable deviation from the first strategy profile, the same player also has a budget-feasible profitable deviation from the second strategy profile. Hence, if the second strategy profile is to be an equilibrium, the first one must be too.

Proposition 5. $E(G^I) \subseteq E(G^{III}_{p^*})$

Proof: Let s^* denote an equilibrium strategy profile of G^I, hence it must be budget-feasible. Let $t^* = T(s^*)$ in $G^{III}_{p^*}$ be as defined above. I claim that this must be an equilibrium of $G^{III}_{p^*}$. First observe that because s^* is budget-feasible, so is t^*, by construction. Hence, by proposition 1 above, the asset markets must clear, whatever p^*. Now I proceed by contradiction, for given p^*.

Assume an agent, say i, can profitably deviate from t^* with a budget-feasible strategy τ^i.

$$\tau^i = ((\beta^i_s, \gamma^i_s); (\overline{\beta}^i_{k,s}, \overline{\gamma}^i_{k,s})^m_{k=1})^r_{s=1}$$

and let $\overline{\pi}_{k,s}$ denote the new price, where

$$\overline{\pi}_{k,s} = \begin{cases} \frac{\sum_{j\neq i} \overline{b}^j_{k,s} + \overline{\beta}^i_{k,s}}{\sum_{j\neq i} \overline{q}^j_{k,s} + \overline{\gamma}^i_{k,s}} & \text{if } \sum_{j\neq i} \overline{q}^j_{k,s} + \overline{\gamma}^i_{k,s} > 0, \\ 0 & \text{else,} \end{cases}$$

where $\overline{q}^j_{k,s}$ and $\overline{b}^j_{k,s}$ are as defined by $T(s^*)$. For τ^i to be budget-feasible, it must be that

$$\sum_k (\overline{\beta}^i_{k,s} - \overline{\pi}_{k,s}\overline{\gamma}^i_{k,s}) \leq \frac{\beta^i_s}{p^*_s} - \gamma^i_s$$

and

$$\sum_s (\beta^i_s - p^*_s \gamma^i_s) \leq 0.$$

Finally note that this deviation yields the following final allocation for individual i:

$$\overline{\zeta^i_{k,s}} = \begin{cases} e^i_{k,s} + \frac{\overline{\beta}^i_{k,s}}{\overline{\pi}_{k,s}} - \overline{\gamma}^i_{k,s} & \text{if } \overline{\pi}_{k,s} > 0, \\ e^i_{k,s} - \overline{\gamma}^i_{k,s} & \text{else.} \end{cases}$$

Now I point to a budget-feasible deviation from s^{i*} in G^I which leads to the same final allocation for i in G^I as did τ^i in $G^{III}_{p^*}$. Let $\sigma^i = S^i(\tau^i, t^{*-i})$, and observe that:

$$\begin{aligned} \gamma^i_{k,s} &= \overline{\gamma}^i_{k,s}, \\ \beta^i_{k,s} &= p^*_s \overline{\beta}^i_{k,s}. \end{aligned}$$

Therefore,

$$\pi_{k,s} = p^*_s \overline{\pi}_{k,s},$$

and hence

$$\zeta^i_{k,s} = \begin{cases} e^i_{k,s} + \frac{\beta^i_{k,s}}{\pi_{k,s}} - \gamma^i_{k,s} &= \overline{\zeta^i_{k,s}} & \text{if } \pi_{k,s} > 0, \\ e^i_{k,s} - \gamma^i_{k,s} &= \overline{\zeta^i_{k,s}} & \text{if } \pi_{k,s} = 0. \end{cases}$$

Hence, if it is budget-feasible, σ^i would be a profitable deviation for individual i. To check the budget constraints, observe that

$$\sum_k (\beta^i_{k,s} - \pi_{k,s}\gamma^i_{k,s}) = p^*_s \sum_k (\overline{\beta}^i_{k,s} - \overline{\pi}_{k,s}\overline{\gamma}^i_{k,s}) \leq p^*_s (\frac{\beta^i_s}{p^*_s} - \gamma^i_s),$$

by budget-feasibility of τ^i, and hence

$$\sum_s \sum_k (\beta^i_{k,s} - \pi_{k,s}\gamma^i_{k,s}) \;\leq\; \sum_s (\beta^i_s - p^*_s\gamma^i_s) \leq 0,$$

also by budget-feasibility of τ^i.

This contradicts the notion that s^* was an equilibrium of G^I and shows that there could not have been a budget-feasible profitable deviation τ^i in $G^{III}_{p^*}$. Therefore, if s^* is an equilibrium of G^I, $T(s^*)$ must be an equilibrium of $G^{III}_{p^*}$ with an identical final allocation and this completes the proof. ■

In the next proposition, I establish the reverse inclusion by the same technique.

Proposition 6. $E(G^{III}_{p^*}) \subseteq E(G^I)$

Proof: Let t^* denote an equilibrium of $G^{III}_{p^*}$ and let $s^* = S(t^*)$. I now show by contradiction that s^* must be an equilibrium of G^I. Assume agent i has a profitable deviation σ^i.

$$\sigma^i = ((\beta^i_{k,s}, \gamma^i_{k,s})^m_{k=1})^r_{s=1}$$

and let $\pi_{k,s}$ denote the new price, where

$$\pi_{k,s} = \frac{\sum_{j\neq i} b^j_{k,s} + \beta^i_{k,s}}{\sum_{j\neq i} q^j_{k,s} + \gamma^i_{k,s}} = \frac{p^*_s \sum_{j\neq i} \overline{b}^j_{k,s} + \beta^i_{k,s}}{\sum_{j\neq i} \overline{q}^j_{k,s} + \gamma^i_{k,s}},$$

if $\sum_{j\neq i} \overline{q}^j_{k,s} + \gamma^i_{k,s} > 0$ and 0 otherwise. Budget-feasibility of σ^i implies that

$$\sum_s \sum_k (\beta^i_{k,s} - \pi_{k,s}\gamma^i_{k,s}) \leq 0 \quad \forall i.$$

Note that this deviation yields the following final allocation for individual i:

$$\zeta^i_{k,s} = \begin{cases} e^i_{k,s} + \frac{\beta^i_{k,s}}{\pi_{k,s}} - \gamma^i_{k,s} & \text{if } \pi_{k,s} > 0, \\ e^i_{k,s} - \gamma^i_{k,s} & \text{else.} \end{cases}$$

I now exhibit a budget-feasible profitable deviation from t^* for individual i in $G^{III}_{p^*}$. This establishes a contradiction and therefore invalidates the claim that $S(t^*)$ was not an equilibrium of G^I. Let $\tau^i = T^i(\sigma^i, s^{-i*})$, i.e.

$$\begin{aligned} \overline{\gamma}^i_{k,s} &= \gamma^i_{k,s}, \forall k, s, \\ \overline{\beta}^i_{k,s} &= \frac{\beta^i_{k,s}}{p^*_s}, \forall k, s, \\ \beta^i_s &= \sum_k \beta^i_{k,s}, \forall s, \\ \gamma^i_s &= \frac{1}{p^*_s} \sum_k \pi_{k,s}\gamma^i_{k,s}, \forall s, \\ \text{hence } \overline{\pi}_{k,s} &= \frac{\pi_{k,s}}{p^*_s}, \forall k, s, \\ \text{and } \overline{\zeta^i_{k,s}} &= \zeta^i_{k,s}, \forall k, s. \end{aligned}$$

The crux of the argument here lies in the fact that β^i_s and γ^i_s can be modified freely without affecting the price of security s. This is where the argument would have failed if it had been applied to G^I and G^{II}.

If τ^i is budget-feasible, it is a profitable deviation since it leads to $\overline{\zeta^i} = \zeta^i$ and

$$\mathcal{U}^i(\zeta^i) > \mathcal{U}^i(\xi^i) = \mathcal{U}^i(\overline{\xi^i}).$$

To check budget-feasibility, simply observe that

$$\sum_k (\overline{\beta}^i_{k,s} - \overline{\pi}_{k,s}\overline{\gamma}^i_{k,s}) = \frac{1}{p^*_s}\sum_k (\beta^i_{k,s} - \pi_{k,s}\gamma^i_{k,s}) = \frac{\beta^i_s}{p^*_s} - \gamma^i_s$$

by definition, and

$$\sum_s (\beta^i_s - p^*_s\gamma^i_s) = \sum_s \sum_k (\beta^i_{k,s} - \pi_{k,s}\gamma^i_{k,s}) \leq 0,$$

by budget-feasibility of σ^i.

Hence if $s^* = S(t^*)$ is not an equilibrium of G^I, t^* can not be an equilibrium of $G^{III}_{p^*}$. This contradicts the assumption and thus completes the proof. ■

Now observe that if $p^*_s = 0$ for some s, the above argument fails precisely in its decisive part. Indeed, if $p^*_s = 0$ then $Q^i_s = 0$ as well and there is no way to transfer income in or out of state s. Hence if the proposed deviation in G^I were to involve such a transfer, it could invalidate s^* as an equilibrium without invalidating t^*. Nonetheless, G^I may have equilibria which endogenously "separate" one or more states. Proposition 7 showed that those can be obtained as equilibria of $G^{III}_{p^*}$ even with positive security prices. Indeed, a positive security price does not force trade. It merely makes it possible.

In terms of the maximization problems, if utilities are differentiable, it is easy to see that at an interior equilibrium

$$\frac{d\mathcal{L}^{III}_i}{db^i_s} = 0 \Leftrightarrow \lambda^i_s = p^*_s\mu^i$$

and looking back at the comparison between G^I and G^{II} it is easy to see that this puts no constraints on the bids and offers on the securities markets. The necessary conditions for equilibria of G^I and $G^{III}_{p^*}$ hence coincide when the proposed equilibrium strategies are linked as in $S(t)$ and $T(s)$ above. The analysis in this section shows that those necessary conditions are also sufficient and extends the result to equilibria which are not interior. Also if there is only one state, the two games coincide trivially.

Note that the only assumption used in the proofs of this section is that there are at least 2 individuals. More noteworthy is the fact that as was the case in the previous section, the proofs in this section do not depend on having more than one good per state. This may seem surprising at first. Indeed, a market game with only one good and one state is in fact not a game, there is nothing to trade. If several such non-games are linked by competitive markets, one might be tempted to think that the resulting game is in fact a competitive economy. In that case, the preceding results could not possibly go through because G^I would be a strategic market game and $G^{III}_{p^*}$ would be a competitive economy. To see why this is not a valid reasoning, simply observe that linking the states by competitive markets allows transfers of wealth from state to state. These transfers affect the relative prices of the goods, the same way that they would in G^I.

6 Concluding Remarks

1. It was my original intention to examine this question closer to the spirit of Shapley and Shubik. I started with a contingent commodity model with commodity money and an Arrow-securities model where the securities were in fact commodity securities and the money was also a commodity money. With these models I hit a first reason why equivalence would not hold. These two models were not payoff equivalent because of liquidity constraints that played out differently in the two different structures. Hence an equilibrium of one game might not be an equilibrium of the other, simply because it is not achievable. I tried to keep the cash-in-advance feature but again, I did not get payoff-equivalence because of liquidity constraints. It seems important to isolate strategic reasons why equilibria of the two games might not be equivalent from feasibility and budget-feasibility considerations.

2. The three games described above have the same sets of budget-feasible allocations. Moreover, the set of achievable allocations of $G^{III}_{p^*}$ contains the corresponding sets for G^I and G^{II}. Hence the differences in equilibrium sets point to differences in strategic power. In particular, the fact that G^I and $G^{III}_{p^*}$ are equivalent whereas G^I and G^{II} aren't highlights the importance of the way asset markets are thought to be organized. It is a fairly common assumption that asset markets are competitive and in this line of thought one could say that the equivalence observed by Arrow and Debreu does hold under this particular form of imperfect competition. If on the other hand one wants to take seriously the fact that the same group of a small number of individuals is involved both on the securities and the goods markets, then one is left with a near non-equivalence.

3. The limited-equivalence result does have one restrictive assumption. I compare only identical final allocations which involve identical *offers* of the goods as well. This may seem disturbing or incomplete but it helps to observe that the equivalence result between G^I and $G^{III}_{p^*}$ is obtained with strategy profiles involving the same offers of all goods in both games as well. The fact that I look only at interior equilibria is less disturbing in that hitting boundary conditions, which are likely to play out differently in the two set-ups, would only make it less likely that the equilibria would coincide.

4. It is important to notice that although there is a sequential aspect to the two-stage games, the equilibrium concept was still Nash equilibrium of the corresponding normal form. Taking the sequential nature of the games seriously poses some problems. For example, existence of subgame perfect equilibria for these games is, to the best of my knowledge, an open question.

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